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CORRELATION SURFACES OF TWO OR MORE INDICES WHEN THE COMPONENTS OF THE INDICES ARE NORMALLY DISTRIBUTED

By George A. Baker

Indices are widely used in statistical analyses.[1] In many cases incorrect conclusions are drawn because indices are not uncorrelated or independent even though all of the component variables are independent. In a previous paper[2] the distribution of an index both of whose components follow the normal law was given exactly i.e. without approximation. The purpose of the present paper is to give the simultaneous distribution of two or more indices when each of the components follow the normal law. The case for two indices will be discussed in detail and the extension to more indices will be indicated.

Let x_1, x_2, and x_3, be correlated variables each being normally distributed about their respective means m_1, m_2, m_3, with standard deviations σ_1, σ_2, σ_3, and let the correlations between the variables in pairs be represented by r_{12}, r_{13}, r_{23}. Then the simultaneous distribution of these three variables will be

$$\begin{aligned}\frac{1}{(2\pi)^{\frac{3}{2}}R^{\frac{1}{2}}\sigma_1\sigma_2\sigma_3}\exp. -\frac{1}{2}\frac{1}{R}\Bigg[&\frac{R_{11}(x_1-m_1)^2}{\sigma_1^2}+\frac{R_{22}(x_2-m_2)^2}{\sigma_2^2}+\frac{R_{33}(x_3-m_3)^2}{\sigma_3^2}\\ &+2R_{12}\frac{(x_1-m_1)(x_2-m_2)}{\sigma_1\sigma_2}+2R_{13}\frac{(x_1-m_1)(x_3-m_3)}{\sigma_1\sigma_3}\\ &+2R_{23}\frac{(x_2-m_2)(x_3-m_3)}{\sigma_2\sigma_3}\Bigg]dx_1dx_2dx_3 \end{aligned}\tag{1}$$

where

$$R=\begin{vmatrix}1 & r_{12} & r_{13}\\ r_{12} & 1 & r_{23}\\ r_{13} & r_{23} & 1\end{vmatrix}$$

and R_{ij} are the respective second order minors of R.

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[1] Rietz, H. L. "On the Frequency Distribution of Certain Ratios," Annals of Mathematical Statistics, Vol. VII, No. 3, Sept. 1936, pp. 145–153.

[2] Baker, G. A., "Distribution of the Means Divided by the Standard Deviations of Samples From Non-homogeneous Populations," Annals of Mathematical Statistics, Feb. 1932, pp. 3–5.

If we make the transformation

$$z_1 = \frac{x_1}{x_3}, \qquad x_1 = z_1 z_3$$

$$z_2 = \frac{x_2}{x_3}, \qquad x_2 = z_2 z_3$$

$$z_3 = x_3, \qquad x_3 = z_3$$

$$dx_1\, dx_2\, dx_3 = z_3^2\, dz_1\, dz_2\, dz_3$$

which is certainly valid if x_1, x_2, x_3, are all positive, then (1) becomes

$$\frac{1}{(2\pi)^{3/2} R^{1/2} \sigma_1 \sigma_2 \sigma_3} \exp. -\frac{1}{2}\frac{1}{R}\left[\frac{R_{11}(z_1 z_3 - m_1)^2}{\sigma_1^2} + \frac{R_{22}(z_2 z_3 - m_2)^2}{\sigma_2^2}\right.$$

$$(2) \qquad + \frac{R_{33}(z_3 - m_3)^2}{\sigma_3^2} + 2R_{12}\frac{(z_1 z_3 - m_1)(z_2 z_3 - m_2)}{\sigma_1 \sigma_2} + 2R_{13}\frac{(z_1 z_3 - m_1)(z_3 - m_3)}{\sigma_1 \sigma_3}$$

$$\left. + 2R_{23}\frac{(z_2 z_3 - m_2)(z_3 - m_3)}{\sigma_2 \sigma_3}\right] z_3^2\, dz_1\, dz_2\, dz_3 .$$

If x_1, x_2, x_3 are all positive the corresponding distribution of z_1 and z_2 can be obtained by integrating (2) between the limits 0 and ∞ with respect to z_3. If x_1, x_2, x_3 are all negative z_1 and z_2 are again both positive so that in order to get the total distribution for z_1 and z_2 it is necessary to add to the integral of (2) between the limits 0 and ∞ with respect to z_3 the similar integral of (2) with z_3 replaced by $-z_3$. The result is

$$(3) \qquad \frac{2e^{-\frac{1}{2}\frac{c}{R}} e^{\frac{1}{2}\frac{b^2}{Ra}}}{(2\pi)^{3/2} R^{1/2} \sigma_1 \sigma_2 \sigma_3}\left[\frac{\sqrt{\pi}}{\sqrt{2}}\frac{R^{3/2}}{a^{3/2}} - \frac{b^2}{a^2}\int_0^{\frac{b}{\sqrt{R}\sqrt{a}}} e^{-\frac{1}{2}z^2}\, dz + \frac{R^{1/2} b^2}{a^{5/2}}\frac{\sqrt{\pi}}{\sqrt{2}}\right]$$

where

$$a = \frac{R_{11}}{\sigma_1^2} z_1^2 + \frac{R_{22}}{\sigma_2^2} z_2^2 + \frac{R_{33}}{\sigma_3^2} + \frac{2R_{12}}{\sigma_1 \sigma_2} z_1 z_2 + \frac{2R_{13}}{\sigma_1 \sigma_3} z_1 + \frac{2R_{23}}{\sigma_2 \sigma_3} z_2$$

$$b = \frac{R_{11}}{\sigma_1^2} m_1 z_1 + \frac{R_{22}}{\sigma_2^2} m_2 z_2 + \frac{R_{33}}{\sigma_3^2} m_3 + \frac{R_{12}}{\sigma_1 \sigma_2} z_1 m_2 + \frac{R_{12}}{\sigma_1 \sigma_2} m_1 z_2 + \frac{R_{13}}{\sigma_1 \sigma_3} m_3 z_1$$

$$+ \frac{R_{13}}{\sigma_1 \sigma_3} m_1 + \frac{R_{23}}{\sigma_2 \sigma_3} m_3 z_2 + \frac{R_{23}}{\sigma_2 \sigma_3} m_2$$

$$c = \frac{R_{11}}{\sigma_1^2} m_1^2 + \frac{R_{22}}{\sigma_2^2} m_2^2 + \frac{R_{33}}{\sigma_3^2} m_3^2 + \frac{2R_{12}}{\sigma_1 \sigma_2} m_1 m_2 + \frac{2R_{13}}{\sigma_1 \sigma_3} m_1 m_3 + \frac{2R_{23}}{\sigma_2 \sigma_3} m_2 m_3 .$$

The same result (3) is obtained for z_1, and z_2 negative, z_1 positive and z_2 negative, z_1 negative and z_2 positive. That is (3) is the simultaneous distribution of z_1 and z_2. The extension to more than 2 indices is immediate. The form of the distribution of the indices and the denominator variable is the same as (2)

except that a, b, and c, the coefficients of z_3^2, z_3 and the constant term respectively in the exponent of e, will be different in that they will include the new indices and the exponent on the denominator variable will be the same as the number of indices involved. The distribution of the indices will again be obtained by integrating from 0 to ∞ with respect to the denominator variable.

The case when all of the variables x_1, x_2, x_3 are independent is especially interesting. If r_{12}, r_{13}, r_{23} are all zero then $R = R_{11} = R_{22} = R_{33} = 1$, $R_{12} = R_{13} = R_{23} = 0$ and a, b, c, become a', b', c', respectively.

$$a' = \frac{z_1^2}{\sigma_1^2} + \frac{z_2^2}{\sigma_2^2} + \frac{1}{\sigma_3^2}$$

$$b' = \frac{m_1 z_1}{\sigma_1^2} + \frac{m_2 z_2}{\sigma_2^2} + \frac{m_3}{\sigma_3^2}$$

$$c' = \frac{m_1^2}{\sigma_1^2} + \frac{m_2^2}{\sigma_2^2} \frac{m_3^2}{\sigma_3^2}$$

Under these conditions and the further condition that m_1, m_2, m_3 are large with respect to σ_1, σ_2, σ_3 respectively so that the integral term of (3) maybe neglected (3) becomes

$$(4) \qquad \frac{e^{-\frac{1}{2}\left(\frac{m_1^2}{\sigma_1^2}+\frac{m_2^2}{\sigma_2^2}+\frac{m_3^2}{\sigma_3^2}\right)}}{2\pi\sigma_1\sigma_2\sigma_3} \frac{e^{\frac{1}{2}\frac{\left(\frac{m_1 z_1}{\sigma_1^2}+\frac{m_2 z_2}{\sigma_2^2}+\frac{m_3}{\sigma_3^2}\right)^2}{\left(\frac{z_1^2}{\sigma_1^2}+\frac{z_2^2}{\sigma_2^2}+\frac{1}{\sigma_3^2}\right)}}}{\left(\frac{z_1^2}{\sigma_1^2}+\frac{z_2^2}{\sigma_2^2}+\frac{1}{\sigma_3^2}\right)^{\frac{3}{2}}} \left[1 + \frac{\left(\frac{m_1 z_1}{\sigma_1^2}+\frac{m_2 z_2}{\sigma_2^2}+\frac{m_3}{\sigma_3^2}\right)^2}{\left(\frac{z_1^2}{\sigma_1^2}+\frac{z_2^2}{\sigma_2^2}+\frac{1}{\sigma_3^2}\right)}\right]$$

It is clear that z_1 and z_2 are not independent in the probability sense for distribution (4).

The question as to the possibility of having the variables independent and the indices independent at the same time arises. Denote the distribution functions of x_1, x_2, x_3, by $X_1(x_1)$, $X_2(x_2)$, $X_3(x_3)$ and of z_1, z_2 by $Z_1(z_1)$, $Z_2(z_2)$. Then, if $x_i \geq 0$, $i = 1, 2, 3$ it is necessary that

$$(5) \qquad \int_a^b X_1(z_3 z_1) X_2(z_3 z_2) X_3(z_3) z_3^2 \, dz_3 = Z_1(z_1) Z_2(z_2)$$

a and b being suitable limits.

For instance, let

$$X_1(x_1) = \frac{1}{x_1^2}, \qquad 1 \leq x_1 \leq 3$$

$$X_2(x_2) = \frac{1}{x_2^2}, \qquad 1 \leq x_2 \leq 3$$

$$X_3(x_3^2) = x_3^2, \qquad 1 \leq x_3 \leq 2$$

then

$$Z_1(z_1) = \frac{c_1}{z_1^2}$$

$$Z_2(z_2) = \frac{c_2}{z_2^2}$$

for value of z_1 and z_2 within a straight line sided area the corners of which are $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 1)$, $(1, 1)$ and $(1, 2)$. z_1, and z_2 are not uncorrelated throughout their entire set of values but are for this particular set of values. Thus is appears that it is possible that the indices may be independent when the variables are, but not necessarily so.

Indices should be used with care since it is very easy to draw invalid conclusions from the consideration of them. Usually it is better to use partial correlation analysis to remove the influence of a third factor than to calculate indices.

THE TYPE B GRAM-CHARLIER SERIES

By Leo A. Aroian

While much attention has been devoted to the Type A Gram-Charlier series for the graduation of frequency curves, the Type B series has been somewhat neglected. However the numerical examples to be presented later will show that the Type B series is very useful for the graduation of skew frequency curves. Wicksell[1] has demonstrated that the Gram-Charlier series may be developed from the same law of probability which forms the basis of the Pearson system of frequency curves. Rietz[2] following Wicksell gives a derivation of the Gram-Charlier series based on the binomial $(q + p)^n$. Jordan[3] gives a method for fitting Type B based on certain orthogonal polynomials which he calls G. He uses factorial moments because of the resulting ease in finding the values of the constants.

We shall consider the Type B series for a distribution of equally distanced ordinates at non-negative values of x. We shall find the values of the first few terms of the series and shall also show how the values of later coefficients may easily be found. We write the Type B series in the form

$$F(x) = c_0 + c_1\Delta\psi(x) + c_2\Delta^2\psi(x) + c_3\Delta^3\psi(x) + c_4\Delta^4\psi(x) + c_5\Delta^5\psi(x) + c_6\Delta^6\psi(x) \tag{1}$$

where

$$\begin{aligned} \psi(x) &= \frac{e^{-m} m^x}{x!}, \qquad m = \mu_1', \text{ the mean,} \\ \Delta\psi(x) &= \psi(x) - \psi(x-1) \quad \text{for } x = 0, 1, 2, \cdots s. \end{aligned} \tag{2}$$

Let $f(x)$ give the ordinates of the observed distribution of relative frequencies, so that $\Sigma f(x) = 1$. To determine the coefficients $c_0, c_1, c_2, \cdots, c_6$, we have, using the method of moments,

$$\begin{aligned}
&\Sigma[c_0\psi(x) + c_1\Delta\psi(x) + c_2\Delta^2\psi(x) + c_3\Delta^3\psi(x) + \ldots\ldots + c_6\Delta^6\psi(x)] = \Sigma f(x) = 1. \\
&\Sigma x[c_0\psi(x) + c_1\Delta\psi(x) + \ldots\ldots\ldots\ldots + c_6\Delta^6\psi(x)] = \Sigma x f(x) = m. \\
&\Sigma x^2[c_0\psi(x) + c_1\Delta\psi(x) + \ldots\ldots\ldots\ldots + c_6\Delta^6\psi(x)] = \Sigma x^2 f(x) = \mu_2'. \\
&\Sigma x^3[c_0\psi(x) + \ldots\ldots\ldots\ldots\ldots\ldots + c_6\Delta^6\psi(x)] = \Sigma x^3 f(x) = \mu_3'. \\
&\Sigma x^4[c_0\psi(x) + \ldots\ldots\ldots\ldots\ldots\ldots + c_6\Delta^6\psi(x)] = \Sigma x^4 f(x) = \mu_4'. \\
&\Sigma x^5[c_0\psi(x) + \ldots\ldots\ldots\ldots\ldots\ldots + c_6\Delta^6\psi(x)] = \Sigma x^5 f(x) = \mu_5'. \\
&\Sigma x^6[c_0\psi(x) + \ldots\ldots\ldots\ldots\ldots\ldots + c_6\Delta^6\psi(x)] = \Sigma x^6 f(x) = \mu_6'.
\end{aligned} \tag{3}$$

Hence we must find the values of

$$\sum_{x=0}^{x=s} x^n \Delta^p \psi(x), \qquad \begin{array}{l} n = 0, 1, 2, 3 \cdots \\ p = 0, 1, 2, 3 \cdots \end{array} \tag{4}$$

defining $\Delta^0\psi(x) = \psi(x)$. We assume that we are dealing with distributions in which s is large, and that the error involved in substituting $\sum_{x=0}^{\infty} x^n\Delta^p\psi(x)$ for $\sum_{x=0}^{s} x^n\Delta^p\psi(x)$ is negligible. To find these summations in a straightforward manner would involve too much labor, so we shall briefly discuss some properties of the generating function, $\psi(x) = \dfrac{e^{-m}m^x}{x!}$, the Poisson exponential, very useful in the graduation of frequency distributions of rare events. The first eight moments about the origin are:

$$\begin{aligned}
&\mu_0' = 1 = \Sigma\psi(x), \qquad \mu_1' = m = \Sigma x\psi(x), \qquad \mu_2' = m + m^2 = \Sigma x^2\psi(x)\\
&\mu_3' = m + 3m^2 + m^3 = \Sigma x^3\psi(x)\\
&\mu_4' = m + 7m^2 + 6m^3 + m^4 = \Sigma x^4\psi(x)\\
&\mu_5' = m + 15\, m^2 + 25m^3 + 10m^3 + m^5 = \Sigma x^5\psi(x)\\
&\mu_6' = m + 31m^2 + 90m^3 + 65m^4 + 15m^5 + m^6 = \Sigma x^6\psi(x)\\
&\mu_7' = m + 63m^2 + 301m^3 + 350\, m^4 + 140m^5 + 21m^6 + m^7 = \Sigma x^7\psi(x)\\
&\mu_8' = m + 127m^2 + 966m^3 + 1701m^4 + 1050m^5 + 256m^6 + 28m^7 + m^8\\
&\qquad = \Sigma x^8\psi(x)
\end{aligned} \tag{5}$$

These may be found by the formula given by Jordan,[3]

$$\mu_{s+1}' = m\left(\mu_s' + \frac{d\mu_s'}{dm}\right). \tag{6}$$

Proof:
$$\frac{d\psi(x)}{dm} = \frac{x\psi(x)}{m} - \psi(x).$$

We multiply by x^n and sum, giving (6). This result may readily be proved also by means of recursion formulas without differentiation. Now we must find the values of

$$\sum_{0}^{\infty} x^n \Delta^p \psi(x) \qquad \begin{array}{l} n = 0, 1, 2, \cdots \\ p = 1, 2, 3, \cdots \end{array}$$

We do this by proving

$$\sum_{x=0}^{x=\infty} x^n \Delta^{s+1}\psi(x) = -\frac{d}{dm}\sum_{x=0}^{x=\infty} x^n \Delta^s \psi(x). \tag{7}$$

Now

$$\frac{d\psi(x)}{dm} = \psi(x-1) - \psi(x) = -\Delta\psi(x). \tag{8}$$

Hence

$$\frac{d}{dm}\Delta^s\psi(x) = \frac{d}{dm}\left[\psi(x) - \binom{s}{1}\psi(x-1) + \binom{s}{2}\psi(x-2) + \cdots + (-1)^s\psi(x-s)\right],$$

since $\Delta^s\psi(x) = \psi(x) - \binom{s}{1}\psi(x-1) + \binom{s}{2}\psi(x-2) + \cdots + (-1)^s\psi(x-s).$

Then by (8)

$$\frac{d}{dm}\Delta^s\psi(x) = \left[\psi(x-1) - \psi(x) - \binom{s}{1}\psi(x-2) + \binom{s}{1}\psi(x-1) + \binom{s}{2}\psi(x-3) - \binom{s}{2}\psi(x-2) + \cdots + (-1)^s\psi(x-s-1) - (-1)^s\psi(x-s)\right].$$

$$\begin{aligned}\frac{d}{dm}\Delta^s\psi(x) &= -\psi(x) + \binom{s+1}{1}\psi(x-1) - \binom{s+1}{2}\psi(x-2) + \cdots - (-1)^s\psi(x-s-1).\\ &= -\left[\psi(x) - \binom{s+1}{1}\psi(x-1) + \binom{s+1}{2}\psi(x-2) + \cdots + (-1)^s\psi(x-s-1)\right].\\ &= -\Delta^{s+1}\psi(x).\end{aligned} \tag{9}$$

We multiply (9) by x^n, sum with respect to x, giving (7).

Thus by use of (7) and (5) we get:

$$\begin{aligned}&\Sigma\Delta^p\psi(x) = 0, \quad p = 1, 2, 3, \cdots\\ &\Sigma x\Delta\psi(x) = -\frac{dm}{dm} = -1.\\ &\Sigma x^2\Delta\psi(x) = -\frac{d}{dm}\Sigma x^2\psi(x) = -\frac{d}{dm}(m+m^2) = -2m-1.\\ &\Sigma x^3\Delta\psi(x) = -3m^2 - 6m - 1.\\ &\Sigma x^4\Delta\psi(x) = -4m^3 - 18m^2 - 14m - 1.\\ &\Sigma x^5\Delta\psi(x) = -5m^4 - 40m^3 - 75m^2 - 30m - 1.\end{aligned} \tag{10}$$

$$\Sigma x^6\Delta\psi(x) = -6m^5 - 75m^4 - 260m^3 - 270m^2 - 62m - 1.$$
$$\Sigma x\Delta^2\psi(x) = 0, \quad \Sigma x^2\Delta^2\psi(x) = 2, \quad \Sigma x^3\Delta^2\psi(x) = 6m + 6.$$
$$\Sigma x^4\Delta^2\psi(x) = 12m^2 + 36m + 14.$$
$$\Sigma x^5\Delta^2\psi(x) = 20m^3 + 120m^2 + 150m + 30.$$
$$\Sigma x^6\Delta^2\psi(x) = 30m^4 + 300m^3 + 780\, m^2 + 540m + 62.$$
$$\Sigma x\Delta^3\psi(x) = 0, \quad \Sigma x^2\Delta^3\psi(x) = 0, \quad \Sigma x^3\Delta^3\psi(x) = -6.$$
$$\Sigma x^4\Delta^3\psi(x) = -24m - 36, \quad \Sigma x^5\Delta^3\psi(x) = -60m^2 - 240m - 150.$$
$$\Sigma x^6\Delta^3\psi(x) = -120m^3 - 900m^2 - 1560m - 540.$$
$$(10) \quad \Sigma x\Delta^4\psi(x) = 0, \quad \Sigma x^2\Delta^4\psi(x) = 0, \quad \Sigma x^4\Delta^4\psi(x) = 24.$$
$$\Sigma x^5\Delta^4\psi(x) = 120m + 240, \quad \Sigma x^3\Delta^4\psi(x) = 0.$$
$$\Sigma x^6\Delta^4\psi(x) = 360m^2 + 1800m + 1560.$$
$$\Sigma x\Delta^5\psi(x) = 0, \quad \Sigma x\Delta^6\psi(x) = 0.$$
$$\Sigma x^2\Delta^5\psi(x) = 0, \quad \Sigma x^2\Delta^6\psi(x) = 0.$$
$$\Sigma x^3\Delta^5\psi(x) = 0, \quad \Sigma x^3\Delta^6\psi(x) = 0.$$
$$\Sigma x^4\Delta^5\psi(x) = 0, \quad \Sigma x^4\Delta^6\psi(x) = 0.$$
$$\Sigma x^5\Delta^5\psi(x) = -120, \quad \Sigma x^5\Delta^6\psi(x) = 0.$$
$$\Sigma x^6\Delta^5\psi(x) = -720m - 1800, \quad \Sigma x^6\Delta^6\psi(x) = 720.$$

Finally we substitute from (5) and (10) into (3), and for μ_n' we substitute $\mu_n' = \sum_{r=0}^{n} \binom{n}{r} \mu_{n-r} m^r$. Hence

$$c_0 = 1$$
$$c_1 = 0$$
$$c_2 = \tfrac{1}{2}\,(\mu_2 - m).$$
$$(11) \quad c_3 = -\frac{1}{3!}(\mu_3 - 3\mu_2 + 2m).$$
$$c_4 = \frac{1}{4!}[\mu_4 - 6\mu_3 + \mu_2(11 - 6m) + 3m(m - 2)].$$
$$c_5 = -\frac{1}{5!}[\mu_5 - 10\mu_4 - \mu_3(10m - 25) + 50\mu_2\,(m - 1) - 4m(5m - 6)].$$
$$c_6 = \frac{1}{6!}[\mu_6 - 15\mu_5 + \mu_4(85 - 15m) + \mu_3(130m - 225) + \mu_2(45m^2 - 375m + 274) - 15m^3 + 130m^2 - 120m].$$

It may be asked whether criteria may be given as guides for the use of Type B. In general Type B may be tried if either the skewness of the distribution to be

fitted is considerable, $\alpha_3 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} > .6$, or if $m = \mu_2 = \mu_3$ approximately. The latter condition strictly would mean that $\psi(x)$ alone is sufficient for a good graduation, if the fourth moment, μ_4, is not used. The examples which follow are arranged to facilitate comparison with the Pearson system of frequency curves. We have an example each of Type I, III, IV, V, VI, and an example of the normal curve.

Type I. Table 1. Here $\alpha_3 > .6$ although $m \neq \mu_2 \neq \mu_3$. The first four moments, unadjusted, give an excellent fit by Type B, which is not quite as good as Type I. The degrees of freedom, according to Fisher,[4] have been taken into consideration here in applying the x^2 test. The two classes 13, 14, were grouped together for the x^2 test. The actual numerical work is easily done on a calculating machine, although logarithms are necessary to find the value of e^{-m}. This example and the remaining are all taken from Elderton[5] with the exception of Type IV which is from A. Fisher.[6]

Type III. Table 2. The unadjusted moments are used. Here $\alpha_3 = 2.0833 > .6$, and $m = \mu_2$ approximately. The fit by Type B is slightly better than that by Type III. We have for Type III $P(x^2 \geq 12.8) = .007$, $n = 3$, while for Type B, $P(x^2) \geq 9.4 = .025$ $n = 3$. Moreover the standard error of prediction for Type III is 11.2 and for Type B is 7.7.

Type IV. Table 3. The rough moments were used. Although $\alpha_3 = .48 < .6$, Type B gives a fine fit since $m = \mu_2 = \mu_3$ approximately. Here the results are given for Type B using 2, 3, and 4 terms of the series. This was done to show how the distribution changes with the addition of more terms. The superiority of Type B over Type IV is evident. The results for Type IV are taken from the class notes of Professor C. C. Craig.

Type V. Table 4. Using the adjusted moments we have a comparison among Types V, A, and B. While the graduations may seem satisfactory, the x^2 test shows that the fit is poor in each case. The order of merit is Type V, Type B, and then Type A. The negative frequencies which appear in Type B may be due to the use of the adjusted moments. If we use the rough moments, the negative frequencies disappear. On the whole the fit by means of the adjusted moments is superior.

Type VI. Table 5. Type VI using the adjusted moments gives an excellent fit. Even though α_3 is considerable, and $\mu_2 = \mu_3$ approximately, four moments with Type B give a poor fit, and five moments, adjusted, achieve a very small gain. Five moments using the unadjusted moments give some improvement, but the -2 frequency in the first class is objectionable.

Normal Curve. Table 6. The normal curve provides a fine fit. $P(x^2 \geq .9) = .96$, $n = 6$. The first two and the last two classes were grouped together for the test. The fit by Type B is less probable, $P(x^2 \geq 8) = .15$, $n = 5$. Type B has two discrepancies, the negative frequencies, and the fact that the total frequencies (neglecting the -1) is 352. That Type B does so well is in itself quite amazing!

TABLE 1

x	Actual frequency	Frequency computed by Pearson Type I	Frequency given by Type B
0	34	44	42.4
1	145	137	121.3
2	156	149	168.7
3	145	142	156.8
4	123	127	120.5
5	103	108	94.9
6	86	88	82.9
7	71	69	72.2
8	55	51	56.7
9	37	36	38.0
10	21	24	23.1
11	13	14	12.0
12	7	7	5.7
13	3	3	2.4
14	1	1	.9

$m = 4.175$ $\alpha_3 = .712247$ Type I $P(x^2 \geq 4.36) = .88$

$\mu_2 = 7.66237$ $\alpha_4 = 2.95214$ n (number of degrees of freedom) $= 9$

$\mu_3 = 15.1069$ $c_2 = 1.74368$

$\mu_4 = 173.326$ $c_3 = -.078298$ Type B $P(x^2 \geq 9.67) = .37$

$c_4 = +.094592$ $n = 9$

$$F(x) = \psi(x) + 1.74368\, \Delta^2\psi(x) - .078298\, \Delta^3\psi(x) + .094592\, \Delta^4\psi(x).$$

TABLE 2

x	Actual frequency	Frequency computed by Type III	Fequency by Type B
0	44	59	48.1
1	135	111	121.6
2	45	45	58.5
3	12	20	10.4
4	8	9	3.5
5	3	4	4.3
6	1	2	2.9
7	3	1	1.2

$m = 1.33466$ $\alpha_3 = \dfrac{\mu_3}{\mu_2^{3/2}} = 2.0833$ $c_2 = .05356$

$\mu_2 = 1.44179$ $c_3 = -.32510$

$\mu_3 = 3.60662$

$$F(x) = \psi(x) + .05356\Delta^2\psi(x) - .32510\Delta^3\psi(x)$$

TABLE 3

Number of alpha particles from a bar of polonium in intervals of $\frac{1}{8}$ of one minute

x	Frequency	Type IV	Type B 2 terms	Type B 3 terms	Type B 4 terms
0	57	50	49.5	49.0	58.2
1	203	183	201.3	201.0	199.8
2	383	392	403.4	404.3	386.1
3	525	544	532.3	533.8	523.9
4	532	539	520.6	521.5	532.1
5	408	417	402.6	402.5	418.2
6	273	250	254.8	254.4	260.2
7	139	131	137.1	136.7	134.0
8	45	61	64.0	63.9	56.7
9	27	26	26.1	26.2	22.9
10	10	12	9.4	9.6	8.6
11	4	4	3.0	3.1	3.6
12	0	1	.9	.9	1.6
13	1	0	.2	.2	.8
14	1	0	.0	.0	.3

$m = 3.87155 \qquad \alpha_3 = .47844$

$\mu_2 = 3.69477 \qquad \alpha_4 = 3.506536$

$\mu_3 = 3.39791$

$\mu_4 = 47.86888$

$$F(x) = \psi(x) - .08839\Delta^2\psi(x) - .00930\Delta^3\psi(x) + .16810\Delta^4\psi(x).$$

Type B, 4 terms $P(x^2 \geq 4.50) = .72,\ n = 7$

Type IV $P(x^2 \geq 10.8) = .15,\ n = 7$

TABLE 4

Mortality Among Female Nominees

x	Deaths	Elderton Type V	Type A	Type B 2 terms	Type B 3 terms	Type B 5 terms	Type B 5 terms
0	4	4	2	1.4	−6.9	−.4	4.1
1	18	10	15	26.3	7.1	9.4	13.1
2	53	80	78	109.7	100.1	84.6	77.4
3	265	261	235	248.3	268.4	252.3	242.5
4	438	441	426	379.5	418.8	425.9	427.4
5	525	480	521	432.7	461.0	484.0	494.1
6	342	381	411	388.8	388.4	402.6	408.1
7	253	247	225	285.4	263.5	259.0	253.9
8	128	137	107	170.8	145.5	132.2	124.9
9	82	68	66	84.3	68.3	58.6	54.1
10	28	32	44	32.9	28.2	26.2	26.4
11	12	14	22	8.6	11.0	13.9	16.4
12	8	6	8	−.01	4.7	8.2	10.7
13	5	3	2	−2.1	2.1	4.3	5.9
14	1	1	0	−1.5	1.3	2.0	2.5

Adjusted moments:

$m = 5.30435$ $\qquad \alpha_3 = .703564$

$\mu_2 = 3.573345$ $\qquad \alpha_4 = 3.996196$

$\mu_3 = +4.752437$

$\mu_4 = 51.02659$

$\mu_5 = 193.439125$

Rough moments:

$m = 5.30435$

$v_2 = 3.65668$

$v_3 = 4.752437$

$v_4 = 52.85276$

$v_5 = 197.39949$

Type A: $f(t) = \varphi(t) + .117261\,\varphi^3(t) + .041508\varphi^4(t)$

Type B: $F(x) = \psi(x) - .86550\Delta^2\psi(x) - .77352\Delta^3\psi(x) + .02814\Delta^4\psi(x) + .57459\Delta^5\psi(x)$

Using uncorrected moments

Type B: $F(x) = \psi(x) - .82384\Delta^2\psi(x) - .73185\Delta^3\psi(x) + .03192\Delta^4\psi(x) + .94033\Delta^5\psi(x)$

(last column above)

TABLE 5

x	Frequency	Type VI	Type B 4 terms	Type B 5 terms
0	1	1	−9.5	−2.0
1	56	50	83.2	69.9
2	167	168	141.6	143.1
3	98	100	102.3	110.7
4	34	36	41.5	40.2
5	9	10	8.7	4.6
6	2	2	.05	2.0
7	1	.5	−.4	1.0

Corrected moments:	Rough moments:
$m = 2.402174$	$m = 2.402174$
$\mu_2 = .928835$	$\mu_2 = 1.012169$
$\mu_3 = .893096$	$\mu_3 = .893096$
$\mu_4 = 4.088800$	$\mu_4 = 4.313176$
	$\mu_5 = 11.28304$
	$\alpha_3 = .87704$
	$\alpha_4 = 4.2101$

Type B, adjusted moments:

$$F(x) = \psi(x) - .73667\Delta^2\psi(x) - .48516\Delta^3\psi(x) - .06424\Delta^4\psi(x) + .10365\Delta^5\psi(x)$$

*Type B, rough moments:

$$F(x) = \psi(x) - .69805\Delta^2\psi(x) - .44654\Delta^3\psi(x) - .06587\Delta^4\psi(x) + .15165\Delta^5\psi(x)$$

* This is used in last column of above. There is a slight error here, which however will not affect the results materially. The third decimal place may be slightly wrong.

TABLE 6

Normal curve

x	Frequency	Normal curve	Type B
0	.6	.6	2.3
1	2.8	2.7	4.7
2	11.5	10.9	8.7
3	27.7	30.1	25.2
4	59.1	58.4	55.2
5	84.7	80.1	79.5
6	74.1	76.9	80.1
7	50.5	52.2	58.1
8	23.2	25.0	29.7
9	12.2	8.4	8.6
10	1.3	2.4	−.9

Moments corrected:

$m = 5.393443$

$\mu_2 = 2.769635$

$\mu_3 = .029805, \mu_4 = 22.40663$

$\alpha_3 = .0064$

$\alpha_4 = 2.920997$

$$\text{Type B: } F(x) = \psi(x) - 1.3119\Delta^2\psi(x) - .4179\Delta^3\psi(x) + 2.1625\Delta^4\psi(x)$$

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A TEST OF A SAMPLE VARIANCE BASED ON BOTH TAIL ENDS OF THE DISTRIBUTION

By John W. Fertig

With the assistance of Elizabeth A. Proehl[1]

(1) Introduction

In testing the hypothesis, say H_0, that an observed sample E of size N has been drawn from a normal population for which the standard deviation, σ, has a particular value, σ_0, one may form the ratio

$$v = \mathop{S}_{i=1}^{N} (x_i - m)^2/\sigma_0^2 = \frac{Nd^2}{\sigma_0^2} \quad \text{(I)}$$

if the population mean m be known, or

$$v' = \mathop{S}_{i=1}^{N} (x_i - \bar{x})^2/\sigma_0^2 = \frac{Ns^2}{\sigma_0^2} \quad \text{(II)}$$

where $\bar{x}$ is the sample mean, if the population mean be unknown. The probability of obtaining a larger (or smaller) value of v or v' than that observed may readily be obtained from the appropriate tail area of the χ^2 distribution with $n = N$ or $n = (N - 1)$ degrees of freedom respectively. The alternative hypotheses to H_0 concerning the normal populations from which the sample may have been drawn assign different values to σ and form a set of hypotheses, Ω. The members of Ω may be classed according to whether they specify $\sigma > \sigma_0$, or $\sigma < \sigma_0$. The practice of regarding only one tail of the distribution, the upper or lower depending on whether $v > N$ or $v < N$, is tantamount to accepting as admissible alternatives to H_0 only one of the classes of Ω.

The alternatives may sometimes be limited to one class or the other through some a priori knowledge, or the problem may be such that only one of the classes is relevant. However, since this is not generally the case, some method of considering all of the alternatives is needed. When testing hypotheses concerning the mean of the sampled population, the problem is quite simple, since the distribution of means is symmetrical. Thus, the "corresponding" value to any positive deviation, $(\bar{x} - m)$, is the negative deviation of the same magnitude. Merely doubling the tail area pertaining to either of the deviations will serve to take account of both classes of alternatives, i.e., those in which $m > m_0$ and those in which $m < m_0$. The problem is more difficult in the case of v or v',

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since the distribution is not symmetrical. In addition to the value of v or v' pertaining to the observed sample we require a "corresponding" value at the other end of the distribution. The definition of "corresponding" which is accepted will determine the required value. There may be a number of such definitions but not all of these will be equally acceptable. The value of v which delimits an equal tail area specifies one of the possible definitions of "corresponding." Another definition would require that the ordinates at the two values of v be equal.

The Neyman and Pearson Approach. Generalized procedures for testing statistical hypotheses have been elaborated in recent years by J. Neyman and E. S. Pearson (1–5). These have considerable philosophical appeal and will be traced as a basis of solution of the immediate problem. A test of a hypothesis H_0 consists essentially of a rule for rejecting H_0 when the observed sample E falls within a suitable critical region w of the N-dimensioned sample space W, and of accepting H_0 when E falls in $(W - w)$. In testing any hypothesis two types of error may be made:

i) H_0 may be rejected when it is true;

ii) H_0 may be accepted when some alternative hypothesis, H_i, is true.

Errors of the first kind may be considered "equivalent" since, if a true hypothesis is to be rejected, it is immaterial which one is chosen. Furthermore, the first type of error can be controlled through our choice of the size of w, say α. The size of w represents the probability of a sample E being an element of w when the hypothesis H_0 is true. This probability may be designated briefly as $P\{E \epsilon w \mid H_0\}$. Then

$$P\{E \epsilon w \mid H_0\} = \int \cdots \int_w p(E \mid H_0)\, dx_1\, dx_2 \cdots dx_N = \alpha \ldots\ldots \text{(III)}$$

where $p(E \mid H_0)$ is the elementary probability law of the sample when H_0 is true, i.e.,

$$p(E \mid H_0) = p(x_1, x_2, \cdots x_N \mid H_0) \ldots\ldots\ldots\ldots\ldots \text{(IV)}$$

Errors of the second type, however, are not equivalent, since their consequences depend on the difference of the true hypothesis from H_0. The utility of a test of H_0 will depend largely on how it controls the second type of error. Ideally, the selection of a critical region should take into consideration the probabilities á priori of the hypotheses composing Ω. Since these probabilities are generally unknown, tests may be sought which are valid independently of them.

A distinction must be made between simple hypotheses which specify completely the elementary probability law of the sample, $p(E)$, and composite hypotheses which specify the law subject to one or more undetermined parameters.

(2) Simple Hypothesis Concerning Population Variance

A test based on a critical region w_0 may be called independent of the probabilities á priori of the alternative hypotheses if it is more powerful than any other

equivalent test for all of the alternative hypotheses (3). An equivalent test is one based on a region w_1 of the same size, α, i.e.,

$$P\{E \epsilon w_0 \mid H_0\} = P\{E \epsilon w_1 \mid H_0\} = \alpha \ldots\ldots\ldots\ldots\ldots \text{(V)}$$

The power of a test based on any critical region, as w_1, is the probability of its rejecting a hypothesis H_0 when some other hypothesis H_i is true. That is, it is the probability of E falling in w_1 when H_i is true. Denote this power by $P\{E \epsilon w_1 \mid H_i\}$. The greater the power of a test, the smaller the risk of the second type of error. If tests as defined above exist, they minimize the probability of the second type of error. Furthermore, the probability of the first type of error is no larger than α. Neyman and Pearson (2) have designated regions satisfying this definition as Best Critical Regions for testing H_0 with regard to the set Ω. If there is no such Best Critical Region, some compromise region must be chosen.

A necessary and sufficient condition for w_0 to be a Best Critical Region with regard to an alternative H_i is that within w_0

$$p(E \mid H_0) \leq kp(E \mid H_i) \ldots\ldots\ldots\ldots\ldots\ldots \text{(VI)}$$

where k is some constant depending on α. If this inequality is true for any H_i, w_0 will be a Best Critical Region for the set Ω.

Neyman and Pearson (2) have shown that in testing the hypothesis that $\sigma = \sigma_0$, when the population mean m is known, there are two Best Critical regions, one pertaining to the class of alternatives for which $\sigma < \sigma_0$ and defined by $v \leq v_1$, the other to the class $\sigma > \sigma_0$ defined by $v \geq v_2$. v_1 and v_2 are values of v so chosen that the size of the critical region shall be α. Although there is no Best Critical Region for all of the alternatives, the choice of a compromise critical region should still depend on its control of the second source of error, that is, on its power for the various alternatives (4). Such a compromise region may be designated as a Good Critical Region. What is needed is a region w_0 of size α defined by the inequalities $v \leq v_1$ and $v \geq v_2$. If v_1 and v_2 are taken as the values cutting off equal tail areas, then the power of the test will be less than α for some values of σ less than σ_0. For those values of σ, H_0 would be accepted more frequently than if it were true. Thus a first requirement for a Good Critical Region is that its power should nowhere be less than α, the value when H_0 is true. Of all such unbiassed Critical Regions of size α, w_0 should then be selected so that its power is everywhere greater than that of any other equivalent unbiassed region.

Critical Regions sufficiently satisfying the above requirements can often be obtained by stipulating that the first derivative of the power function with respect to θ, the parameter under consideration, shall be zero at $\theta = \theta_0$, and that the second shall be a maximum there. Then not only does the probability of the second source of error decrease as we move away from θ_0, but it decreases most rapidly in the vicinity of θ_0. Critical Regions satisfying these conditions are called unbiassed Critical Regions of Type A, (4). Under certain assumptions

concerning the nature of the elementary probability law $p(E \mid \theta)$ it can be shown that w_0 is defined by the inequalities $\varphi_1 \leq c_1$ and $\varphi_1 \geq c_2$ where c_1 and c_2 satisfy the conditions

$$\int_{c_1}^{c_2} p(\varphi_1)\, d\varphi_1 = 1 - \alpha \quad \text{(VII)}$$

$$\int_{c_1}^{c_2} \varphi_1 p(\varphi_1)\, d\varphi_1 = 0 \quad \text{(VIII)}$$

where
$$\varphi_1 = \left.\frac{d \log p(E \mid \theta)}{d\theta}\right|_{\theta=\theta_0} \quad \text{(IX)}$$

and $p(\varphi_1)$ is the distribution function of φ_1.

In applying these results to the testing of the hypothesis that $\sigma = \sigma_0$[2] when the population mean is known,

$$\varphi_1 = (v - N)/\sigma_0 \quad \text{(X)}$$

Obviously $p(v)$, the distribution of v, may be considered instead of $p(\varphi_1)$. w_0 is defined by the inequalities $v \leq v_1$ and $v \geq v_2$ where

$$\int_0^{v_1} p(v)\, dv + \int_{v_2}^{\infty} p(v)\, dv = \alpha_1 + \alpha_2 = \alpha \quad \text{(XI)}$$

$$\int_{v_1}^{v_2} (v - N)p(v)\, dv = v^{N/2}e^{-v/2}\Big|_{v_1}^{v_2} = 0 \quad \text{(XII)}$$

w_0 so defined is also of type A_1, that is, its power curve lies everywhere above that of any other equivalent region, vanishing in the first derivative at $\sigma = \sigma_0$, (4).

The use of w_0 as the appropriate critical region is equivalent to the use of r as a test criterion, where

$$v^{N/2}e^{-\frac{1}{2}v} = r) \quad \text{XIII)}$$

That is, a value of v yielding the same r as the observed v may be taken as the corresponding value. Reference to the appropriate tables and summing of the two tail areas gives P_r, the probability of obtaining a smaller value of r when H_0 is true. H_0 may be rejected if P_r is less than some previously fixed number, say α. If the distribution of r could be evaluated the necessity of dealing with two values of v would be obviated.

The criterion r is equivalent to that deduced by the use of maximum likelihood ratios (6). Thus,

$$p(E \mid \sigma^2) = (2\pi\sigma^2)^{-N/2} e^{-\overset{N}{\underset{i=1}{S}}(x_i - m)^2/2\sigma^2} \quad \text{(XIV)}$$

[2] The solution is the same in terms of σ^2.

Maximizing $p(E \mid \sigma^2)$ for fixed E and all possible σ^2 we have

$$p_{\max.}(E \mid \sigma^2) = N^{N/2}\left[2\pi \mathop{S}_{i=1}^{N} (x_i - m)^2\right]^{-N/2} e^{-N/2} \quad \text{.........(XV)}$$

$$\lambda = \frac{p(E \mid \sigma_0^2)}{p_{\max.}(E \mid \sigma^2)} = N^{-N/2} v^{N/2} e^{-\frac{1}{2}(v-N)} \quad \text{.............(XVI)}$$

$$= N^{-N/2} e^{N/2} r \quad \text{..........................(XVII)}$$

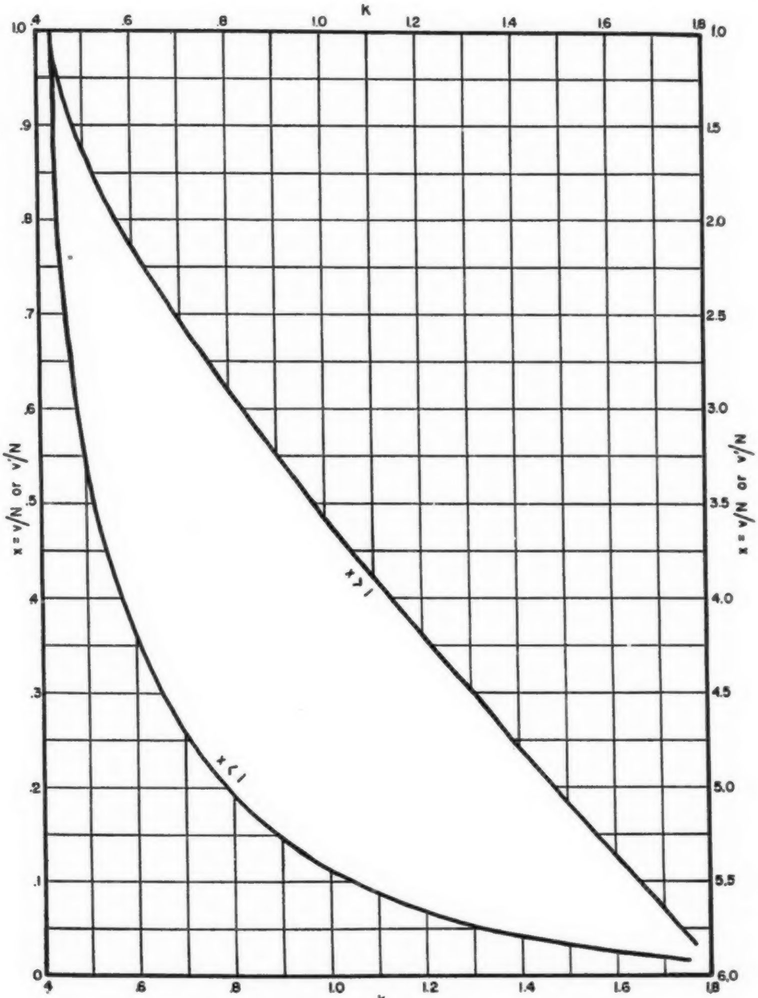

FIG. 1. Graph of Equation $x - \log_e x = k \log_e 10$

The h^{th} moment coefficient of λ about zero, $\mu_h'(\lambda)$, is given by

$$\mu_h'(\lambda) = \frac{\Gamma\left[\dfrac{N(1+h)}{2}\right]}{\Gamma(N/2)} (2e/N)^{hN/2} (1+h)^{-N(1+h)/2} \quad \text{......(XVIII)}$$

TABLE I

Probability that a sample has been drawn from a normal population with a specified variance or standard deviation

Degrees of Freedom, n

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0.435	.9724	.9581	.9473	.9383	.9305	.9235	.9171	.9111	.9055	.9002	.8952	.8905	.8859	.8815	.8773	.8732	.8693	.8655	.8617	.8581	.8546	.8512	.8478	.8445	.8413
0.440	.9217	.8812	.8509	.8259	.8042	.7848	.7671	.7508	.7357	.7215	.7081	.6954	.6833	.6717	.6606	.6500	.6398	.6299	.6204	.6112	.6023	.5936	.5853	.5771	.5692
0.445	.8928	.8377	.7968	.7632	.7341	.7083	.6850	.6636	.6438	.6253	.6080	.5917	.5762	.5616	.5476	.5343	.5215	.5093	.4975	.4862	.4753	.4649	.4547	.4450	.4355
0.450	.8704	.8041	.7552	.7151	.6808	.6505	.6232	.5983	.5754	.5542	.5344	.5159	.4984	.4820	.4664	.4516	.4375	.4241	.4113	.3990	.3873	.3760	.3652	.3549	.3449
0.455	.8513	.7758	.7203	.6752	.6367	.6029	.5726	.5452	.5202	.4971	.4756	.4557	.4370	.4195	.4030	.3874	.3726	.3587	.3454	.3328	.3208	.3094	.2985	.2881	.2781
0.460	.8346	.7510	.6899	.6405	.5987	.5621	.5296	.5003	.4737	.4492	.4267	.4059	.3865	.3683	.3514	.3355	.3205	.3064	.2931	.2806	.2687	.2574	.2467	.2366	.2269
0.465	.8194	.7287	.6628	.6098	.5651	.5263	.4920	.4613	.4335	.4082	.3850	.3636	.3439	.3255	.3084	.2925	.2776	.2637	.2506	.2383	.2267	.2158	.2055	.1958	.1866
0.470	.8055	.7083	.6382	.5821	.5350	.4944	.4587	.4270	.3984	.3725	.3489	.3273	.3074	.2890	.2721	.2563	.2417	.2281	.2154	.2035	.1924	.1819	.1722	.1630	.1544
0.475	.7926	.6896	.6156	.5568	.5077	.4657	.4289	.3963	.3672	.3410	.3172	.2956	.2758	.2576	.2409	.2255	.2113	.1981	.1859	.1745	.1639	.1541	.1449	.1364	.1284
0.480	.7805	.6721	.5946	.5335	.4827	.4395	.4019	.3688	.3393	.3130	.2892	.2677	.2481	.2303	.2140	.1990	.1853	.1726	.1610	.1502	.1402	.1310	.1224	.1145	.1071
0.485	.7692	.6557	.5751	.5119	.4597	.4155	.3773	.3438	.3142	.2879	.2643	.2430	.2238	.2064	.1906	.1761	.1630	.1509	.1398	.1296	.1203	.1117	.1037	.0964	.0897
0.490	.7583	.6402	.5569	.4918	.4384	.3934	.3547	.3211	.2915	.2653	.2420	.2211	.2023	.1854	.1701	.1562	.1436	.1322	.1217	.1122	.1035	.0955	.0881	.0814	.0753
0.495	.7481	.6256	.5397	.4729	.4185	.3729	.3340	.3003	.2708	.2449	.2219	.2015	.1832	.1668	.1521	.1388	.1269	.1160	.1062	.0973	.0892	.0818	.0750	.0689	.0633
0.500	.7382	.6117	.5234	.4552	.4000	.3539	.3148	.2812	.2519	.2263	.2038	.1838	.1661	.1503	.1362	.1236	.1122	.1020	.0928	.0845	.0770	.0702	.0640	.0584	.0534
0.510	.7197	.5857	.4933	.4228	.3663	.3197	.2806	.2473	.2188	.1940	.1725	.1537	.1372	.1226	.1097	.0983	.0882	.0792	.0712	.0640	.0577	.0519	.0468	.0422	.0381
0.520	.7025	.5619	.4660	.3937	.3364	.2897	.2509	.2183	.1907	.1670	.1466	.1290	.1138	.1004	.0888	.0786	.0697	.0618	.0549	.0488	.0434	.0386	.0344	.0307	.0273
0.530	.6864	.5399	.4411	.3674	.3097	.2632	.2250	.1933	.1667	.1442	.1251	.1087	.0947	.0826	.0721	.0631	.0553	.0484	.0425	.0373	.0328	.0289	.0254	.0224	.0197
0.540	.6713	.5194	.4181	.3435	.2856	.2396	.2023	.1716	.1461	.1248	.1070	.0918	.0790	.0681	.0588	.0508	.0439	.0381	.0330	.0286	.0249	.0216	.0188	.0164	.0143
0.550	.6570	.5002	.3969	.3216	.2639	.2186	.1822	.1526	.1284	.1083	.0917	.0778	.0661	.0563	.0480	.0410	.0351	.0300	.0257	.0221	.0189	.0163	.0140	.0120	.0103
0.560	.6434	.4822	.3772	.3015	.2442	.1997	.1643	.1360	.1130	.0942	.0787	.0660	.0554	.0466	.0393	.0332	.0280	.0237	.0201	.0170	.0144	.0123	.0104	.0089	.0075
0.570	.6304	.4652	.3588	.2830	.2263	.1827	.1485	.1213	.0996	.0820	.0678	.0561	.0466	.0387	.0322	.0269	.0224	.0188	.0157	.0132	.0110	.0093	.0078	.0065	.0055
0.580	.6180	.4492	.3417	.2659	.2099	.1673	.1343	.1084	.0879	.0715	.0584	.0478	.0392	.0322	.0265	.0218	.0180	.0149	.0123	.0102	.0084	.0070	.0058	.0048	.0040
0.590	.6061	.4340	.3256	.2501	.1949	.1534	.1217	.0970	.0777	.0625	.0504	.0407	.0330	.0268	.0218	.0177	.0145	.0118	.0097	.0079	.0065	.0053	.0044	.0036	.0029
0.600	.5946	.4195	.3105	.2354	.1811	.1408	.1103	.0869	.0688	.0546	.0435	.0348	.0278	.0223	.0180	.0145	.0117	.0094	.0076	.0061	.0050	.0040	.0033	.0027	.0022
0.610	.5836	.4057	.2963	.2217	.1685	.1294	.1001	.0779	.0609	.0478	.0376	.0297	.0235	.0186	.0148	.0118	.0094	.0075	.0060	.0048	.0038	.0031	.0025	.0020	.0016
0.620	.5730	.3926	.2829	.2090	.1568	.1190	.0909	.0699	.0540	.0419	.0326	.0254	.0199	.0156	.0122	.0096	.0076	.0060	.0047	.0037	.0029	.0023	.0018	.0015	.0012
0.630	.5627	.3801	.2702	.1971	.1461	.1094	.0826	.0628	.0479	.0367	.0282	.0218	.0168	.0130	.0101	.0079	.0061	.0048	.0037	.0029	.0023	.0018	.0014	.0011	.0009
0.640	.5528	.3682	.2583	.1860	.1362	.1008	.0752	.0564	.0426	.0322	.0245	.0187	.0143	.0109	.0084	.0064	.0049	.0038	.0029	.0023	.0018	.0014	.0010	.0008	.0006

0.650	.5432	.3567	.2470	.1757	.1270	.0928	.0684	.0508	.0378	.0283	.0213	.0160	.0121	.0091	.0069	.0053	.0040	.0030	.0023	.0018	.0014	.0010	.0008	.0006	.0005
0.660	.5339	.3457	.2363	.1659	.1185	.0856	.0623	.0457	.0336	.0249	.0185	.0137	.0103	.0077	.0057	.0043	.0032	.0024	.0018	.0014	.0010	.0008	.0006	.0005	.0003
0.670	.5249	.3352	.2261	.1568	.1106	.0789	.0568	.0411	.0299	.0219	.0160	.0118	.0087	.0064	.0048	.0035	.0026	.0020	.0015	.0011	.0008	.0006	.0005	.0003	.0003
0.680	.5161	.3251	.2165	.1483	.1033	.0728	.0518	.0371	.0267	.0193	.0140	.0101	.0074	.0054	.0040	.0029	.0021	.0016	.0012	.0008	.0006	.0005	.0003	.0003	.0002
0.690	.5076	.3154	.2073	.1403	.0965	.0672	.0472	.0334	.0237	.0169	.0121	.0087	.0063	.0045	.0033	.0024	.0017	.0013	.0009	.0007	.0005	.0004	.0003	.0002	.0001
0.700	.4993	.3060	.1986	.1327	.0902	.0621	.0431	.0301	.0212	.0149	.0106	.0075	.0053	.0038	.0027	.0020	.0014	.0010	.0007	.0005	.0004	.0003	.0002	.0001	.0001
0.750	.4609	.2642	.1610	.1011	.0647	.0419	.0274	.0181	.0120	.0080	.0053	.0036	.0024	.0016	.0011	.0007	.0005	.0003	.0002	.0002	.0001	.0001	.0000	.0000	.0000
0.800	.4268	.2292	.1312	.0776	.0468	.0286	.0176	.0109	.0068	.0043	.0027	.0017	.0011	.0007	.0004	.0003	.0002	.0001	.0001	.0000	.0000	.0000			
0.850	.3962	.1995	.1074	.0598	.0339	.0195	.0114	.0067	.0039	.0023	.0014	.0008	.0005	.0003	.0002	.0001	.0001	.0000	.0000						
0.900	.3686	.1742	.0883	.0463	.0248	.0134	.0074	.0041	.0023	.0013	.0007	.0004	.0002	.0001	.0001	.0000	.0000								
0.950	.3435	.1525	.0727	.0359	.0181	.0093	.0048	.0025	.0013	.0007	.0004	.0002	.0001	.0001	.0000										
1.000	.3205	.1338	.0601	.0280	.0133	.0064	.0031	.0015	.0008	.0004	.0002	.0001	.0000	.0000											
1.050	.2994	.1175	.0497	.0218	.0098	.0045	.0020	.0010	.0004	.0002	.0001	.0000													
1.100	.2800	.1034	.0412	.0170	.0072	.0031	.0013	.0006	.0003	.0001	.0001														
1.150	.2621	.0911	.0342	.0133	.0053	.0022	.0009	.0004	.0002	.0001	.0000														
1.200	.2455	.0803	.0284	.0105	.0039	.0015	.0006	.0002	.0001	.0000															
1.250	.2301	.0709	.0236	.0082	.0029	.0011	.0004	.0001	.0001																
1.300	.2158	.0626	.0197	.0064	.0022	.0007	.0003	.0001	.0000																
1.350	.2024	.0553	.0164	.0051	.0016	.0005	.0002	.0001																	
1.400	.1900	.0490	.0137	.0040	.0012	.0004	.0001	.0000																	
1.450	.1785	.0433	.0114	.0031	.0009	.0003	.0001																		
1.500	.1677	.0384	.0096	.0025	.0007	.0002	.0000																		
1.800	.1159	.0187	.0033	.0006	.0001	.0000																			
2.100	.0807	.0092	.0011	.0001	.0000																				
2.400	.0564	.0045	.0004	.0000																					
2.700	.0395	.0022	.0001																						
3.000	.0278	.0011	.0000																						

TABLE I—*Concluded*

k	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
0.435	.8382	.8351	.8321	.8291	.8262	.8234	.8206	.8178	.8151	.8124	.8098	.8072	.8046	.8021	.7996	.7972	.7947	.7923	.7900	.7876	.7853	.7831	.7808	.7786	.7764
0.440	.5614	.5539	.5466	.5394	.5324	.5256	.5189	.5124	.5060	.4998	.4936	.4876	.4818	.4760	.4704	.4648	.4594	.4540	.4488	.4437	.4386	.4336	.4287	.4239	.4192
0.445	.4263	.4175	.4089	.4005	.3924	.3846	.3769	.3695	.3623	.3553	.3484	.3417	.3352	.3289	.3227	.3167	.3108	.3051	.2995	.2940	.2887	.2835	.2784	.2734	.2685
0.450	.3353	.3261	.3172	.3086	.3003	.2923	.2845	.2771	.2698	.2628	.2561	.2495	.2432	.2370	.2310	.2252	.2196	.2142	.2089	.2037	.1988	.1939	.1892	.1846	.1802
0.455	.2686	.2595	.2507	.2424	.2343	.2266	.2192	.2120	.2052	.1986	.1922	.1861	.1802	.1745	.1691	.1638	.1587	.1538	.1490	.1445	.1401	.1358	.1317	.1277	.1238
0.460	.2177	.2090	.2006	.1927	.1851	.1779	.1710	.1644	.1580	.1520	.1462	.1407	.1354	.1303	.1254	.1208	.1163	.1120	.1079	.1039	.1002	.0965	.0930	.0896	.0864
0.465	.1779	.1697	.1619	.1545	.1475	.1409	.1346	.1286	.1229	.1174	.1123	.1074	.1027	.0982	.0940	.0899	.0861	.0824	.0789	.0755	.0723	.0693	.0664	.0636	.0609
0.470	.1463	.1387	.1315	.1247	.1183	.1123	.1066	.1012	.0962	.0914	.0868	.0825	.0784	.0746	.0709	.0675	.0642	.0611	.0581	.0553	.0527	.0501	.0477	.0455	.0433
0.475	.1209	.1139	.1073	.1012	.0954	.0900	.0849	.0801	.0757	.0714	.0675	.0638	.0602	.0569	.0538	.0509	.0481	.0455	.0431	.0407	.0386	.0365	.0345	.0327	.0310
0.480	.1002	.0939	.0879	.0824	.0772	.0724	.0679	.0637	.0598	.0561	.0527	.0495	.0465	.0437	.0410	.0386	.0362	.0341	.0320	.0301	.0284	.0267	.0251	.0236	.0222
0.485	.0834	.0776	.0723	.0673	.0627	.0584	.0545	.0508	.0474	.0442	.0413	.0385	.0360	.0336	.0314	.0293	.0274	.0256	.0239	.0224	.0209	.0196	.0183	.0171	.0160
0.490	.0696	.0644	.0596	.0551	.0511	.0473	.0439	.0406	.0377	.0350	.0324	.0301	.0279	.0259	.0241	.0224	.0208	.0193	.0179	.0167	.0155	.0144	.0134	.0125	.0116
0.495	.0582	.0535	.0492	.0453	.0417	.0384	.0354	.0326	.0300	.0277	.0256	.0236	.0218	.0201	.0185	.0171	.0158	.0146	.0135	.0125	.0115	.0106	.0098	.0091	.0084
0.500	.0487	.0446	.0407	.0373	.0341	.0312	.0286	.0262	.0240	.0220	.0202	.0185	.0170	.0156	.0143	.0131	.0120	.0111	.0102	.0093	.0086	.0079	.0072	.0067	.0061
0.510	.0344	.0311	.0281	.0254	.0229	.0208	.0188	.0170	.0154	.0140	.0126	.0115	.0104	.0094	.0085	.0078	.0070	.0064	.0058	.0053	.0048	.0043	.0039	.0036	.0033
0.520	.0244	.0218	.0194	.0174	.0155	.0139	.0124	.0111	.0099	.0089	.0080	.0071	.0064	.0057	.0051	.0046	.0041	.0037	.0033	.0030	.0027	.0024	.0022	.0019	.0017
0.530	.0174	.0153	.0135	.0119	.0106	.0093	.0082	.0073	.0064	.0057	.0051	.0045	.0040	.0035	.0031	.0028	.0024	.0022	.0019	.0017	.0015	.0013	.0012	.0011	.0009
0.540	.0124	.0108	.0095	.0082	.0072	.0063	.0055	.0048	.0042	.0037	.0032	.0028	.0025	.0022	.0019	.0017	.0014	.0013	.0011	.0010	.0009	.0008	.0007	.0006	.0005
0.550	.0089	.0077	.0066	.0057	.0049	.0043	.0037	.0032	.0027	.0024	.0021	.0018	.0015	.0013	.0011	.0010	.0009	.0007	.0006	.0006	.0005	.0004	.0004	.0003	.0003
0.560	.0064	.0055	.0047	.0040	.0034	.0029	.0025	.0021	.0018	.0015	.0013	.0011	.0010	.0008	.0007	.0006	.0005	.0004	.0004	.0003	.0003	.0002	.0002	.0002	.0001
0.570	.0046	.0039	.0033	.0028	.0023	.0020	.0017	.0014	.0012	.0010	.0008	.0007	.0006	.0005	.0004	.0004	.0003	.0003	.0002	.0002	.0002	.0001	.0001	.0001	.0001
0.580	.0033	.0028	.0023	.0019	.0016	.0013	.0011	.0009	.0008	.0006	.0005	.0005	.0004	.0003	.0003	.0002	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0000
0.590	.0024	.0020	.0016	.0013	.0011	.0009	.0008	.0006	.0005	.0004	.0003	.0003	.0002	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0000	.0000	.0000	
0.600	.0017	.0014	.0012	.0009	.0008	.0006	.0005	.0004	.0003	.0003	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0000	.0000	.0000				
0.610	.0013	.0010	.0008	.0007	.0005	.0004	.0003	.0003	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0000	.0000	.0000							
0.620	.0009	.0007	.0006	.0005	.0004	.0003	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0000	.0000										
0.630	.0007	.0005	.0004	.0003	.0003	.0002	.0002	.0001	.0001	.0001	.0001	.0000	.0000												
0.640	.0005	.0004	.0003	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0000														
0.650	.0004	.0003	.0002	.0002	.0001	.0001	.0001	.0001	.0000	.0000															
0.660	.0003	.0002	.0001	.0001	.0001	.0001	.0000	.0000																	
0.670	.0002	.0001	.0001	.0001	.0001	.0000																			
0.680	.0001	.0001	.0001	.0001	.0000																				
0.690	.0001	.0001	.0001	.0000																					
0.700	.0001	.0001	.0000																						
0.750	.0000	.0000																							

For N infinite, $(-2\log_e\lambda)$ will be distributed as χ^2 with one degree of freedom. For finite values of N, however, we have not been able to evaluate the distribution of λ, although the distribution of the Incomplete Beta Function serves as a good approximation. Approximate distributions for several values of N have been obtained. P_λ, the probability of obtaining a smaller value of λ than that observed, as obtained from these distributions agrees well with the sum of the tail areas pertaining to v_1 and v_2 yielding the same value of λ (or r). The construction of tables is simplified by taking (1)

$$\log_{10}\lambda = N/2(\log_{10} e - k) \quad \text{(XIX)}$$

That is,

$$x - \log_e x = k \log_e 10 \quad \text{(XX)}$$

where $x = v/N$. Equation (XX) is independent of N and may be solved once and for all for x, given k.[3] In Figure 1 is plotted the graph of equation (XX). For convenience, the branch of the curve giving the roots greater than unity has been folded back with altered scale from the minimum value of k, $\log_{10}e$, occurring at $x = 1$. Table I was then constructed by multiplying the two values of x for a given k by $(N/2)^{\frac{1}{2}}$, referring to the Tables of the Incomplete Gamma Function (7) with $p = (N - 2)/2$, and adding the resulting two tail areas. The values for the odd numbers above 12 were obtained by interpolating between the even numbers. For $N = 1$, $(x)^{\frac{1}{2}}$ was used as a normal deviate. The values in Table I should be correct to four decimals. Table I is entered with the number of degrees of freedom, n, on which x is based. In the case of the simple hypothesis this is N.

The following may serve as an illustration: Blood urea nitrogen determinations (mg./100 cc.) were made on a sample of 25 schizophrenic patients. The mean was found to be 15.56, the variance, 10.486. Previous investigation of blood urea nitrogen on a large sample of normal control subjects gave a mean of 16.03 and a variance of 20.268, which for the purpose of the example may be considered as the population parameters. Then we may wish to test the hypothesis that the variance of the sampled population, σ^2, is $\sigma_0^2 = 20.268$, knowing the mean of the sampled population to be 16.03. Calculate

$$x = \frac{s^2 + (\bar{x} - m)^2}{\sigma_0^2} = .528$$

Referring to Fig. 1, the value of k is about .505. Turning to Table I with $k = .505$,[4] $n = 25$, P is found to be .0457. We should thus be inclined to reject the hypothesis.

For N small, the area of the tail of the distribution near zero is considerably larger than that at the upper end. As N increases the distribution of v becomes

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[3] If the solution were explicit the distribution of λ could easily be deduced from that of x.

[4] k obtained directly from (XX) is .507, corresponding to $P = .0427$.

more and more symmetrical and the two areas approach equality. Even for $N = 50$, however, they are rather unequal, so that merely doubling the area pertaining to the observed v does not give a sufficiently accurate approximation. For $N > 50$ an approximation correct within several units in the third decimal place may be obtained by taking $\sqrt{2N}(\sqrt{x} - 1)$ as a normal deviate. This assumes that the standard deviation is normally distributed with variance $\sigma_0^2/2N$.

(3) Composite Hypothesis Concerning Population Variance

Here H_0 specifies only the value of the parameter $\theta = \theta_0$, leaving undetermined the value of a second parameter, ν. Thus, H_0 consists of a subset, ω, of simple hypotheses, each of which specifies a different value for ν. Any simple hypothesis specifying different values of both parameters, θ and ν, is an alternative to H_0. These alternatives form the set Ω. The elementary probability law determined by H_0 is $p(E \mid H_0) = p(E \mid \theta_0\nu)$, while that determined by an alternative hypothesis H_i is $p(E \mid H_i) = p(E \mid \theta_i\nu_i)$. In testing composite hypotheses the first requirement is to find regions "similar" to W with regard to ν, i.e., such that the chance of rejection of a true hypothesis, $P\{E \epsilon w \mid H_0\}$, equals α for all the values of ν specified by the simple hypotheses composing H_0. A test based on a similar region w_0 may be called independent of the probabilities á priori, if its power with respect to all the alternatives of Ω is greater than that of any other similar region w_1 of the same size, α, (3). Let

$$\varphi_2 = \partial \log p(E \mid \theta\nu)/\partial\nu \big|_{\theta=\theta_0} \dots\dots\dots\dots\dots\dots \text{(XXI)}$$

Then the equations $\varphi_2 =$ constant will describe hypersurfaces in N-dimensioned space, on one of which the observed E must fall. Under certain assumptions pertaining to the law of elementary probability it can be shown (2) that a necessary and sufficient condition for w to be a similar region is that

$$P\{E \epsilon w(\varphi_2) \mid H_0\} = \alpha P\{E \epsilon W(\varphi_2) \mid H_0\} \dots\dots\dots\dots \text{(XXII)}$$

for all values of φ_2, where $w(\varphi_2)$ and $W(\varphi_2)$ are parts of the surface $\varphi_2 =$ constant common to w and W respectively. A similar region is then built up of these parts $w(\varphi_2)$ obtaining for the various values of φ_2. The Best Critical Region, w_0, for a particular simple alternative, H_i, must then be composed of pieces, $w_0(\varphi_2)$, maximizing $P\{E \epsilon w_0(\varphi_2) \mid H_i\}$. The problem is the same as for simple hypotheses except that we shall be working in a space $W(\varphi_2)$ of $(N - 1)$ dimensions. $w_0(\varphi_2)$ is defined by the inequality

$$p(E \mid H_i) \geq k(\varphi_2)\ p(E \mid H_0) \dots\dots\dots\dots\dots\dots \text{(XXIII)}$$

where $k(\varphi_2)$ is some constant depending on α. If $w_0(\varphi_2)$ is the same for all H_i, then w_0 is the Best Critical Region for testing H_0 with respect to Ω.

Neyman and Pearson showed (2) that in testing the composite hypothesis that $\sigma = \sigma_0$ when the population mean is unknown there are two Best Critical Regions corresponding to the class of alternatives $\sigma < \sigma_0$ and $\sigma > \sigma_0$, defined respectively by the inequalities $v' \leq v_1'$ and $v' \geq v_2'$. If the whole set of alternatives, Ω, is to

be considered some compromise region must be sought. Dealing with the case where similar regions exist Neyman (5) defines a Critical Region as unbiassed and of Type B if the first derivative of the power function, $P(E \epsilon w \mid H_i)$, with respect to θ vanishes at $\theta = \theta_0$, and if the second derivative at that point is a maximum. Let

$$\varphi_1 = \frac{\partial \log p(E \mid \theta v)}{\partial \theta}\bigg|_{\theta=\theta_0} \qquad \text{(XXIV)}$$

Then it can be shown that the desired region will be defined by the inequalities $\varphi_1 \leq k_1(\varphi_2)$ and $\varphi_1 \geq k_2(\varphi_2)$ where $k_1(\varphi_2)$ and $k_2(\varphi_2)$ are determined to satisfy

$$\int_{k_1(\varphi_2)}^{k_2(\varphi_2)} p(\varphi_1\varphi_2)\, d\varphi_1 = (1 - \alpha)p(\varphi_2) \qquad \text{(XXV)}$$

and

$$\int_{k_1(\varphi_2)}^{k_2(\varphi_2)} \varphi_1 p(\varphi_1\varphi_2)\, d\varphi_1 = (1 - \alpha) \int_{-\infty}^{\infty} \varphi_1 p(\varphi_1\varphi_2)\, d\varphi_1 \qquad \text{(XXVI)}$$

where $p(\varphi_2)$ is the distribution function of φ_2, and $p(\varphi_1\varphi_2)$ is the simultaneous distribution of φ_1 and φ_2.

Applying equations (XXV) and (XXVI) it follows that the appropriate Critical Region is defined by the inequalities $v' \leq v_1'$ and $v' \geq v_2'$ where

$$\alpha = \alpha_1 + \alpha_2 = \int_0^{v_1'} p(v')\, dv' + \int_{v_2'}^{\infty} p(v')\, dv' \qquad \text{(XXVII)}$$

and

$$v'^{(N-1)/2} e^{-\frac{1}{2}v'} \bigg|_{v_1'}^{v_2'} = 0 \qquad \text{(XXVIII)}$$

where $p(v')$ is the distribution function of v'.

The use of the unbiassed Critical Region of Type B corresponds to adopting as a criterion

$$v'^{(N-1)/2} e^{-\frac{1}{2}v'} = r' \qquad \text{(XXIX)}$$

Since v' derived from a sample of size N is distributed as v derived from a sample of size $(N - 1)$, it follows that r' is equivalent to the r of equation (XIII) based on a sample of size $(N - 1)$. Therefore Table I may also be used for testing the hypothesis that $\sigma = \sigma_0$ whatever be the population mean, by entering with the number of degrees of freedom, $N - 1$.

In the example previously used, compute

$$x = \frac{s^2}{\sigma_0^2} = 0.517$$

From Figure 1, k is approximately .51, corresponding to $P = .0422$.

r' is not the same as the maximum likelihood ratio λ' (6).

$$\lambda' = \frac{p_{\max}(E \mid \sigma_0^2 m)}{p_{\max}(E \mid \sigma^2 m)} = N^{-N/2} v'^{N/2} e^{-\frac{1}{2}(v'-N)} = N^{-N/2} e^{N/2} v'^{\frac{3}{2}} r' \ldots \text{(XXX)}$$

As N becomes infinite the distribution of λ' is the same as that of the λ of (XVI). For $N = 49$, the probabilities corresponding to λ' agree with those using r' to within a unit in the third decimal.

The λ' test is biassed as may be seen in Figure 2 where we have plotted the power of the test based on the region w defined by $v_1' = 3.187$, $v_2' = 22.912$ for which $\alpha = .0436 + .0064 = .0500$, on the assumption that $\sigma_0^2 = 1.0$, for $N = 10$. Although the criterion is biassed it is slightly more sensitive to alternatives

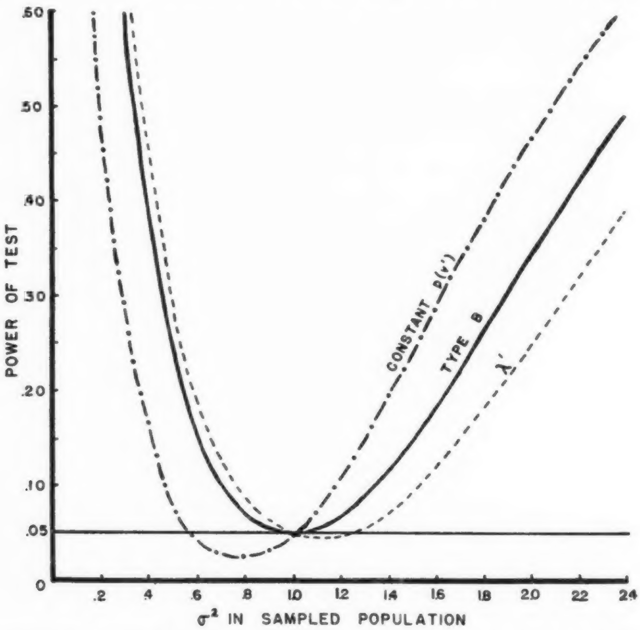

FIG. 2. Comparison of Critical Regions for v'. H_0 Specifies $\sigma_0^2 = 1.0$. $N = 10$.

specifying $\sigma^2 < \sigma_0^2$ than is the unbiassed Critical Region of Type B defined by $v_1' = 2.953$, $v_2' = 20.305$, $\alpha = .0339 + .0161 = .0500$. The criterion of constant distribution, $p(v')$,

$$v'^{(N-3)/2} e^{-\frac{1}{2}v'} = c' \ldots\ldots\ldots\ldots\ldots\ldots \text{(XXXI)}$$

has also been considered. In this case $v_1' = 1.903$, $v_2' = 17.391$, $\alpha = .0071 + .0429 = .0500$. This criterion is biassed for some alternatives specifying $\sigma^2 < \sigma_0^2$, but its power curve lies above that of the unbiassed region for $\sigma^2 > \sigma_0^2$.

Apparently the bias may be shifted at will by changing the exponent of v'. This may be desirable if greater weight is to be given to one class of alternatives. In fact decreasing the exponent of v' to 0 produces the Best Critical Region

for the class of alternatives specifying $\sigma^2 > \sigma_0^2$, and defined by $v_1 = 0$, $v_2 = 16.919$ for $\alpha = .0500$. No region can be found giving greater power. On the other hand this region is insensitive to alternatives of the other class. Increasing the exponent indefinitely produces the Best Critical Region for the other class defined by $v_2' = \infty$ and $v_1' = 3.325$ for $\alpha = .0500$.

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ON THE POLYNOMIALS RELATED TO THE DIFFERENTIAL EQUATION $\frac{1}{y}\frac{dy}{dx} = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2} \equiv \frac{N}{D}$

By Frank S. Beale

Introduction. In a previous issue of this Journal,[1] E. H. Hildebrandt has established the existence of a general system of polynomials $P_n(k, x)$ associated with the solutions of Pearson's Differential Equation

$$\text{(R)} \qquad \frac{1}{y}\frac{dy}{dx} = \frac{N}{D},$$

N and D being polynomials in x of degrees not exceeding one and two respectively with no factor in common.

It was shown that the polynomials $P_n(k, x) \equiv P_n$ themselves satisfy certain differential equations and a recurrence relation. The classical polynomials of Hermite, Legendre, Laguerre, and Jacobi are special types of $P_n(k, x)$. Since the classical polynomials are employed rather extensively in statistical theory, certain of their properties are of special interest.

It is the purpose of this paper to determine from Hildebrandt's general equations some new properties of $P_n(k, x)$ and to apply these properties to the classical polynomials. The paper consists of two parts. In part I some theorems are established concerning common zeros of D and P_n. In particular, a theorem is established to exhibit the conditions under which the zeros of P_n, which are not zeros of D, are simple. In part II a method is outlined for the classical polynomials by which one can determine the number and location of the real zeros in the various segments into which the zeros of D divide the x axis. The points of inflexion and the degree of the polynomials are also considered.

A new feature of the method employed is, we believe, its being based upon the use of differential equations of first order, for most part, while other investigators[2] have employed differential equations of second order. As to the results obtained, the author believes them to be partly new. They have points in common with the results of Fujiwara, Lawton and Webster.

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I. Theorems Concerning Common Zeros of $P_n(k, x)$ and D

The following equations will be employed later:

$$P_{n+1}(k, x) = [N + (k - n)D']P_n(k, x) + DP'_n(k, x). \tag{1}$$

$$P'_{n+1}(k, x) = (n + 1)\left[N' + \frac{2k - n}{2} D''\right] P_n(k, x). \tag{2}$$

$$\begin{aligned} P_{n+1}(k, x) &= [N + (k - n)D']P_n(k, x) \\ &\quad + n\left[N' + \frac{2k - n + 1}{2} D''\right] DP_{n-1}(k, x). \end{aligned} \tag{3}$$

These are not explicitly given in Hildebrandt's Paper but the method of obtaining them is outlined there in detail.

We shall make use of the following lemma which we state without proof.

Lemma (1). *Let $P_n(x)$ be a polynomial of degree n. If both P_n and P'_n contain a factor $(x - \alpha)^m$, $m < n$, then P_n contains the factor $(x - \alpha)^{m+1}$.*

We also need an expression for $P^{(q)}_{n+1}(k, x)$. By repeatedly differentiating (2) and eliminating $P'_n(k, x)$ we get,

$$\begin{aligned} P^{(q)}_{n+1}(k, x) &= \prod_{i=0}^{q-1} (n + 1 - i)\left[N' + \frac{2k - n + i}{2} D''\right] P_{n-q+1}(k, x), \\ & \qquad q = 1, 2, \cdots (n + 1). \end{aligned} \tag{4}$$

Theorem I_1. If D is a perfect square, D′ is not a factor of $P_{n+1}(k, x)$, $n =$ 0, 1, 2, $\cdots$

Proof: Assume D' to be a factor of P_{n+1}. From (1), D' is either a factor of P_n or of $N + (k - n) D'$. But D' is not a factor of $N + (k - n) D'$ as this implies that D' is a factor of N contrary to hypothesis on (R) that D and N have no factor in common. Thus, D' is a factor of P_n, and by a repetition of the reasoning a factor finally of P_1, which as it was just pointed out, is impossible.

Theorem I_2. Set $D = (\alpha_1 x + \beta_1)(\alpha_2 x + \beta_2)$, D not a perfect square. If $\alpha_i x + \beta_i$, $i = 1$ or 2, is a factor of P_n, then $(\alpha_i x + \beta_i)^q$ is a factor of P_{n+q-1}, $q = 1, 2, 3, \cdots$

Proof: From (1), $\alpha_i x + \beta_i$ being a factor of P_n and D, is also a factor of P_{n+1}. From (2), $\alpha_i x + \beta_i$ is a factor of P'_{n+1}. From Lemma (1) it follows that $(\alpha_i x + \beta_i)^2$ is a factor of P_{n+1}. Continued repetition of the reasoning establishes the theorem.

Corollary. If both $\alpha_1 x + \beta_1$ and $\alpha_2 x + \beta_2$ are factors of P_n, then D^q is a factor of P_{n+q-1}.

Theorem I_3. Assume D of the same form as in Theorem I_2. If $\alpha_i x + \beta_i$, $i = 1$ or 2, is a factor of P_{n+1} and no higher power of $\alpha_i x + \beta_i$ is such a factor then $\alpha_i x + \beta_i$ is a factor of $N + (k - n)D'$.

Proof: From (1), $\alpha_i x + \beta_i$ being a factor of P_{n+1} and of D is also a factor of either $N + (k - n)D'$ or of P_n. But $\alpha_i x + \beta_i$ a factor of P_n requires, from I_2, that $(\alpha_i x + \beta_i)^2$ be a factor of P_{n+1} contrary to hypothesis. Thus, $\alpha_i x + \beta_i$ is a factor of $N + (k - n)D'$.

Corollary. If $(\alpha_1 x + \beta_1)(\alpha_2 x + \beta_2)$, $(\alpha_1, \alpha_2 \neq 0)$, *is a factor of* P_{n+1} *and no higher power of either* $\alpha_1 x + \beta_1$ *or* $\alpha_2 x + \beta_2$ *is contained in* P_{n+1} *then* $N + (k - n) D' \equiv 0$. For from I_3, $N + (k - n)D'$ contains $(\alpha_1 x + \beta_1)(\alpha_2 x + \beta_2)$ as a factor which implies $N + (k - n)D'$, being linear, vanishes identically.

Theorem I_4. *If* $(\alpha_i x + \beta_i)^q$ *and no higher power of* $\alpha_i x + \beta_i$ *is a factor of* P_{n+q-1} *then* $\alpha_i x + \beta_i$ *and no higher power of* $\alpha_i x + \beta_i$ *is a factor of* P_n.

Proof: Let us write,

(A) $P_{n+q-1} = (\alpha_i x + \beta_i)^q \phi_{n-1}$, $\phi_{n-1} \equiv$ a polynomial of degree $\leq n - 1$ which does not contain the factor $\alpha_i x + \beta_i$. Taking the $(q - 1)^{st.}$ derivative of (A) by Leibnitz Theorem, we get,

$$\text{(B)} \qquad P^{(q-1)}_{n+q-1} = \sum_{i=0}^{q-1} \binom{q-1}{i} \frac{d^i}{dx^i} (\alpha_i x + \beta_i)^q \frac{d^{q-1-i}}{dx^{q-1-i}} \phi_{n-1}.$$

On setting $q = q - 1$ in (4) there results,

$$\text{(C)} \quad P^{(q-1)}_{n+q-1} = \prod_{i=0}^{q-2} (n + q - 1 - i)\left[N' + \frac{2k - n - q + i + 2}{2} D''\right] P_n.$$

From (B) we see that $\alpha_i x + \beta_i$ is a factor of $P^{(q-1)}_{n+q-1}$. No higher power of $\alpha_i x + \beta_i$ is such a factor. From (C) our theorem now follows.

Corollary (1). *Under the hypotheses of Theorem* I_4, $\alpha_i x + \beta_i$ *is a factor of* $N + (k - n + 1)D'$. This follows at once from I_4 and I_3.

Corollary (2). *If* $D^q = (\alpha_1 x + \beta_1)^q (\alpha_2 x + \beta_2)^q$, $(\alpha_1, \alpha_2 \neq 0)$, *is a factor of* P_{n+q-1} *and no higher powers of either* $\alpha_1 x + \beta_1$ *or* $\alpha_2 x + \beta_2$ *are factors, then* $N + (k - n + 1)D' \equiv 0$. For the linear expression $N + (k - n + 1)D'$ contains, from Corollary (1), the quadratic factor $(\alpha_1 x + \beta_1)(\alpha_2 x + \beta_2)$.

The following lemma can be easily established and is given without proof.

Lemma (2). *Assume D of the same form as in Theorem* I_2. *Then there is only one value of s for which* $N + sD'$ *contains* $\alpha_i x + \beta_i$ *as a factor.*

Theorem I_5. *Assume D of the same form as in Theorem* I_2. *If* $N + (k - n)D'$ *contains* $\alpha_i x + \beta_i$, $i = 1$ *or* 2, *as a factor, then* P_{n+1} *contains* $\alpha_i x + \beta_i$ *and no higher power of* $\alpha_i x + \beta_i$ *as a factor.*

Proof: From (1) we see that P_{n+1} contains $\alpha_i x + \beta_i$ at least to the first power as a factor. Again from (1), if P_{n+1} contains a higher power of $\alpha_i x + \beta_i$ as a factor, this means that both P_n and P_n' contain $\alpha_i x + \beta_i$ at least to the first power as a factor and from Lemma (1) it follows that P_n contains $\alpha_i x + \beta_i$ at least to the second power as a factor. By corollary (1) from Theorem I_4 it follows that $\alpha_i x + \beta_i$ is a factor of $N + (k - n_1)D'$ for $n_1 < n$, contrary to Lemma (2).

Theorem I_6. *If* $\alpha_1 x + \beta_1$ *and* $\alpha_2 x + \beta_2$ *are factors of* $N + (k - n_1)D'$ *and* $N + (k - n_2)D'$ *respectively,* $(\alpha_1, \alpha_2 \neq 0)$, *then* $P_\mu \equiv 0$, $\mu > n_1 + n_2$.

Proof: From Theorems I_5 and I_2 we see that $(\alpha_1 x + \beta_1)^{n_2} (\alpha_2 x + \beta_2)^{n_1}$, of degree $n_1 + n_2$, is a factor of $P_{n_2+n_1}$, of degree $n_2 + n_1$ at most. Similarly,

$(\alpha_1 x + \beta_1)^{n_2+1} (\alpha_2 x + \beta_2)^{n_1+1}$, of degree $n_2 + n_1 + 2$, is a factor of $P_{n_2+n_1+1}$, of degree $n_2 + n_1 + 1$ at most. This implies $P_{n_2+n_1+1} \equiv 0$. Hence, $P_\mu \equiv 0$, $\mu > n_1 + n_2$. In fact, (1) shows that $P_\mu \equiv 0$ implies $P_\nu \equiv 0$, $\nu > \mu$.

Theorem I_7. *Assume D of the same form as in Theorem* I_2. *Then* $P_{n+1} \equiv 0$, $P_n \not\equiv 0$, *implies either* $N + (k - m)D' \equiv 0$, $m \leq n$, *or there exist two values of* m, (m_1, m_2), *such that* $N + (k - m_1)D'$, $N + (k - m_2)D'$ *contain as factors* $\alpha_1 x + \beta_1$ *and* $\alpha_2 x + \beta_2$ *respectively*, $(m_1, m_2 \leq n)$.

Proof: Setting $P_{n+1} \equiv 0$ in (1) gives,

$$(1^0)\ [N + (k - n)D']\, P_n + DP'_n \equiv 0.$$

If $P_n \equiv$ const., 1^0 shows that $N + (k - n)D' \equiv 0$ and our theorem is verified. Suppose $P_n \not\equiv$ const. We get from (1^0),

$$P'_n = -\frac{[N + (k - n)D']P_n}{D}.$$

Thus, D is a factor of the numerator, and our theorem now follows from Corollaries (1) and (2) of Theorem I_4.

Theorem I_8. *If* $N + (k - m)D' \not\equiv 0$, $m = 1, 2, \cdots n$, *and if* $N + (k - m)D'$ *contains neither* $\alpha_1 x + \beta_1$, *nor* $\alpha_2 x + \beta_2$ *as factors, then* P_{n+1} *and D have no factors in common.* This follows at once from Theorems I_2 and I_4 which constitute a necessary and sufficient condition that P_n and D have factors in common.

Theorem I_9. *If* $N \equiv$ *const. and if D is linear, all* P_n *are constants*, $n = 1, 2, 3, \cdots$. This follows directly from (2).

Theorem I_{10}. *If* $N' + \dfrac{2k - m}{2} D'' \not\equiv 0$, $m = 1, 2, \cdots (n - 1)$, *all zeros of* P_n *which are not zeros of D are simple.*

Proof: Suppose P_n has a multiple zero $x = \alpha$ which is not a zero of D. Then (1) shows that α is a zero of P_{n+1}. From (2), α is a zero of P'_{n+1}. From Lemma (1), α is at least a double zero of P_{n+1}. Furthermore, (3) shows that α being a double zero of P_n and of P_{n+1} is also a double zero of P_{n-1}. By a continued application of (3), it follows that α is a double zero of P_1 which is impossible since P_1 is of degree ≤ 1.

II. Concerning the Zeros of $P_n(k, x)$

The polynomials $P_n(k, x)$ are defined by Hildebrandt[3] as follows: $P_n(k, x) = \dfrac{1}{y} D^{n-k} \dfrac{d^n}{dx^n} D^k y$ where y is a non-identically vanishing solution of the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2} \equiv \frac{N}{D}.$$

[3] L.c. pp. 400–401.

The Jacobi Polynomials are defined as follows:

$$J_n(x, \alpha, \beta) = x^{1-\alpha}(1-x)^{1-\beta} \frac{d^n}{dx^n}[x^{n+\alpha-1}(1-x)^{n+\beta-1}], \alpha, \beta$$

real. It follows that $J_n(x, \alpha, \beta)$ is a special type of $P_n(k, x)$ with $N \equiv (-\beta-\alpha)x + \alpha$, $D \equiv x(1-x)$, $n = k + 1$, whence,

$$N' = -\beta-\alpha, \qquad D' = 1 - 2x, \qquad D'' = -2; \qquad D(0) = D(1) = 0,$$

$$P_1(k, x) \equiv N + kD' = 0 \text{ for}$$

$$x = \frac{\alpha + k}{\alpha + \beta + 2k}, \qquad P_1'(k, x) = -\beta - \alpha - 2k.$$

In determining the number and location of the real zeros of the Jacobi Polynomials we employ the following notations:

$P_i(k, x) = 0$ for $x = \alpha_{i,k,j}$, $\quad i = 1, 2, \cdots k + 1; k = 0, 1, 2, \cdots; j = 1, 2, \cdots i.$

$\alpha_{i,k,j} \leq \alpha_{i+1,k,j}$

$$\theta = N' + \frac{2k-n}{2}D'' = -\beta - \alpha - 2k + n, \quad n = 1, 2, \cdots k,$$

$$\mu = [N + (k-n)D']_{x=0} = \alpha + (k-n),$$

$$\nu = [N + (k-n)\, D']_{x=1} = -\beta - (k-n).$$

We proceed to determine the number of real zeros of the Jacobi Polynomials on the intervals $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$ into which the zeros of D divide the x axis.[4] The proofs proceed by mathematical induction. We first determine the location of the real zeros of $P_n(k, x)$, $n = 1, 2, \cdots k + 1$, by successive applications of (1) and (2). We then use the relation $P_{k+1}(k, x) \equiv J_{k+1}(x, \alpha, \beta)$.

Several cases concerning possible values of α and β should be considered. In order to bring out the method of procedure only two such cases will be fully discussed here. The results for other possible cases will be merely listed.

A_1 : $\alpha < 0, \beta < 0, |\alpha| < |\beta|, \alpha, \beta, \alpha + \beta$ *not integers.*

Let k_1 be the greatest integer contained in α,

" k_2 " " " " " " β,

" k_3 be the greatest integral value of k for which $\alpha + \beta + 2k < 0$. Then

$$0 \leq k_1 \leq k_3 \leq k_2.$$

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[4] In the case $\alpha, \beta > 0$ these zeros all lie, as is known, on $(0, 1)$.

$A_{11}: 0 \leq k \leq k_1$. *We then have* $\theta > 0$, $\mu < 0$, $\nu > 0$, $0 < \alpha_{1k1} < 1$, $P_1' > 0$. *Then* $J_{k+1}(x, \alpha, \beta)$ *has* $\dfrac{(1)^k + (-1)^k}{2}$ *zeros in* 0, 1. *These are the only real zeros.*

Proof: Consider first $P_1(k, x)$. Its only zero is at $\alpha_{1,k,1}$, where $0 < \alpha_{1,k,1} < 1$. Furthermore, $P_1' > 0$. Also $P_1 > 0$ for $x > \alpha_{1,k,1}$ and < 0 for $x < \alpha_{1,k,1}$. From (1) we see that $P_2(k, \alpha_{1,k,1}) > 0$, (since $P_1(k, \alpha_{1,k,1}) = 0$, $D(\alpha_{1,k,1}) > 0$ and $P_1' > 0$). From (2) it follows that $P_2'(k, x) < 0$ for $x < \alpha_{1,k,1}$, $P_2'(k, \alpha_{1,k,1}) = 0$, $P_2'(k, x) > 0$ for $x > \alpha_{1,k,1}$. These conclusions follow from remarks concerning the sign of θ, the fact that $P_1(k, \alpha_{1,k,1}) = 0$, and from remarks concerning the sign of P_1 to the left and to the right of $x = \alpha_{1,k,1}$. Thus, $P_2(k, x) > 0$ for all real x and hence has no real zeros. By employing (2), it is now evident that $P_3'(k, x) > 0$. From (1) and remarks concerning μ and ν we see that $P_3(k, 0) < 0$ and $P_3(k, 1) > 0$. Thus $P_3(k, x)$ has a single real zero $\alpha_{3,k,1}$, $0 < \alpha_{3,k,1} < 1$. The reasoning from P_3 to P_4 is analogous to that from P_1 to P_2. By continuing this procedure we finally conclude that $P_{k+1}(k, x)$, ($\equiv J_{k+1}(x, \alpha, \beta)$, has but one real zero, (in 0, 1), if k is even and no real zeros if k is odd.

A_{12}: $k_1 < k \leq k_3$. *Set* $k = k_1 + q$, $q = 1, 2, \cdots, k_3 - k_1$. *Here* $\theta > 0$, $\mu > 0, n = 1, 2, \cdots q - 1, \mu < 0, n = q, q + 1, \cdots, q + k_1$. $\nu > 0$, $\alpha_{1,k,1} < 0$, $P_1'(k, x) > 0$. $J_{k_1} + q + 1\ (x, \alpha, \beta)$ *has* q *distinct zeros in* $(-\infty, 0)$ *and* $\dfrac{(1)^{k_1} + (-1)^{k_1}}{2}$ *zeros in* 0, 1. *These are the only real zeros.*

Proof: First consider the sequence $P_n(k, x)$ $n = 1, 2, \cdots q$, since the conditions on θ, μ, and ν do not change over this range of n. Now $P_1(k, \alpha_{1,k,1}) = 0$, $\alpha_{1,k,1} < 0$. Furthermore since $P_1' > 0$ we have $P_1 > 0$ for $x > \alpha_{1,k,1}$ and < 0 for $x < \alpha_{1,k,1}$. Pass now to $P_2(k, x)$. Since $D(\alpha_{1,k,1}) < 0$ and $P_1'(k, \alpha_{1,k,1}) > 0$, we see from (1) that $P_2(k, \alpha_{1,k,1}) < 0$. Moreover (2) shows $P_2'(k, \alpha_{1,k,1}) = 0$, $P_2'(k, x) < 0$ for $x < \alpha_{1,k,1}$ and > 0 for $x > \alpha_{1,k,1}$. Thus $P_2(k, x) < 0$ and a relative minimum at $x = \alpha_{1,k,1}$. Since $|P_2(k, \pm\infty)| = \infty$, we see that $P_2(k, x)$ has two real zeros of which the left most, $\alpha_{2,k,1}$, is in $(-\infty, 0)$. Again $\mu > 0$ together with (1) assures $P_2(k, 0) \not> 0$. Thus $\alpha_{2,k,2}$ is in $(\alpha_{1,k,1}, 0)$, hence in $(-\infty, 0)$. By continuing this reasoning on the successive $P_n(k, x)$, $n = 1, 2, \cdots q$, we conclude that $P_q(k, x)$ has q zeros in $-\infty$, 0 and $P_q'(k, \alpha_{q,k,1}) < 0$.

Next, consider the sequence $P_n(k, x)$, $n = q + 1, q + 2, \cdots q + k_1 + 1$. Over this range of n we have $\theta > 0$, $\mu < 0$, $\nu > 0$. From what has just been shown, $P_q(k, \alpha_{q,k,i}) = 0$, $-\infty < \alpha_{q,k,i} < 0$, $i = 1, 2, \cdots q$. Also $P_q'(k, \alpha_{q,k,i})$, $i = 1, 2, \cdots q$, is alternately negative and positive. Suppose q odd, (similar reasoning holds for q even). Thus, we suppose $P_q'(k, \alpha_{q,k,1}) < 0$, $P_q'(k, \alpha_{q,k,q}) < 0$, $P_q(k, x) > 0$ for $x < \alpha_{q,k,1}$ and < 0 for $x > \alpha_{q,k,q}$. (1) shows $P_{q+1}(k, \alpha_{q,k,i})$, $i = 1, 2, \cdots q$, to be alternately positive and negative. Thus, the zeros $\alpha_{q,k,i}$ are separated by $q - 1$ zeros of $P_{q+1}(k, x)$. Since from (1), $P_{q+1}(k, \alpha_{q,k,1}) > 0$ and from (2) $P_{q+1}'(k, x) > 0$ for $x < \alpha_{q,k,1}$, there exists a zero $\alpha_{q+1,k,1}$ in $(-\infty, \alpha_{q,k,1})$. Thus far, we have established the existence of q zeros of $P_{q+1}(k, x)$ in $(-\infty, 0)$. q being odd, we have from (1), $P_{q+1}(k, \alpha_{q,k,q}) > 0$. Also from (2),

$P'_{q+1}(k, x) < 0$ for $x > \alpha_{q,k,q}$. Again from (1) and assumptions regarding μ and ν it follows that $P_{q+1}(k, 0) > 0$, $P_{q+1}(k, 1) < 0$. Thus, $P_{q+1}(k, x)$ has a zero $\alpha_{q+1,k,q+1}$ in $(0, 1)$. There being no extrema for $P_{q+1}(k, x)$ other than the $\alpha_{q,k,i}$, $i = 1, 2, \cdots q$, (as (2) shows), we have thus proved that $P_{q+1}(k, x)$ has q distinct zeros in $(-\infty, 0)$ and a single zero in $(0, 1)$. Reasoning similarly from $P_{q+1}(k, x)$ to $P_{q+2}(k, x)$ we establish the existence of q distinct zeros $\alpha_{q+2,k,i}$, $i = 1, 2, \cdots q$, in $(-\infty, 0)$ with $\alpha_{q+2,k,1}$ in $(-\infty, \alpha_{q+1,k,1})$ and $\alpha_{q+2,k,i}$, $i = 2, 3, \cdots q$, separating $\alpha_{q+1,k,i}$, $i = 1, 2, \cdots q$. From (1) we see that $P_{q+2}(k, \alpha_{q+1,k,q}) < 0$ and $P_{q+2}(k, \alpha_{q+1,k,q+1}) < 0$. The only extrema of $P_{q+2}(k, x)$, (as (2) shows), are located at $\alpha_{q+1,k,i}$, $i = 1, 2, \cdots q + 1$. Again, by (2), $P'_{q+2}(k, x) < 0$ for $x > \alpha_{q+1,k,q+1}$; hence there can be no real zeros of P_{q+2} except the q zeros in $(-\infty, 0)$ already found. The reasoning from P_{q+2} to P_{q+3} is similar to that from P_q to P_{q+1}. Thus, $P_{q+k_1+1} \equiv J_{k_1+q+1}$ has q distinct zeros in $(-\infty, 0)$, together with one zero in $(0, 1)$ for k_1 even. For k_1 odd, there are q distinct zeros in $(-\infty, 0)$ only. The results are the same whether q is odd or even.

The results for the remaining sub-cases under case A_1 are given in the table which follows. For completeness, the results for cases A_{11} and A_{12} are included in the tabulation. A few words of explanation are necessary to clarify the conditions under which the various sub-cases in the table occur. Let $|\alpha| = k_1 + q$, $|\beta| = k_2 + h$, $h, q < 1$. If $q + h < 1$, then $|\alpha + \beta| = k_1 + k_2$ and we have either,

A_{131} : $k_1 + k_2$ *even*, $2k_3 = k_1 + k_2 \equiv k_3 - k_1 = k_2 - k_3$.

A_{132} : $k_1 + k_2$ *odd*, $2k_3 = k_1 + k_2 - 1 \equiv k_3 - k_1 = k_2 - k_3 - 1$.

Again if $1 < q + h < 2$, then $|\alpha + \beta| = k_1 + k_2 + 1$ and we have either,

A_{133} : $k_1 + k_2 + 1$ *even*, $2k_3 = k_1 + k_2 + 1 \equiv k_3 - k_1 = k_2 - k_3 + 1$.

A_{134} : $k_1 + k_2 + 1$ *odd*, $2k_3 = k_1 + k_2 \equiv k_3 - k_1 = k_2 - k_3$

In cases A_{141} and A_{151} we assume $|\alpha + \beta| = k_1 + k_2 + p$, $p < 1$, while in cases A_{142} and A_{152}, $|\alpha + \beta| = k_1 + k_2 + p$, $1 < p < 2$. The complete results for case A_1 follow. (See page 213.)

A_2 : $\alpha < 0, \beta < 0, |\alpha| < |\beta|$, α, β *not integers*, $\alpha + \beta =$ *integer*. Define k_1, k_2, k_3 as in A_1. Then $0 \leq k_1 \leq k_3 \leq k_2$. In Case A_{21}, $\beta + \alpha$ is odd while in Case A_{22}, $\beta + \alpha$ is even. (See page 214.)

A_3 : $\alpha < 0, \beta < 0, \alpha = -k_1'$, *integer*, β *not an integer*, $|\alpha| < |\beta|$. Define k_1, k_2, k_3 as in A_1. Then $0 \leq k_1 \leq k_3 \leq k_2$. There are two sub-cases, A_{31} : the greatest integral value of $\alpha + \beta$ is odd, A_{32} : this integral value is even. (See page 215.)

A_4 : $\alpha < 0, \beta < 0$, α *not an integer*, $\beta = -k_1$, *integer*, $|\alpha| < |\beta|$. Define k_1, k_2, k_3 as in A_1. Then $0 \leq k_1 \leq k_3 \leq k_2$. There are two sub-cases, A_{41} : the integral part of $\alpha + \beta$ is odd, A_{42} : this integral value is even. (See page 216).

Cases	Polynomial	Range of Sub-Script	Zeros in $(-\infty, 0)$	Zeros in $(0, 1)$	Zeros in $(1, \infty)$
A_{11}	J_{k+1};	$0 \leq k \leq k_1$;	0;	$\frac{(1)^k + (-1)^k}{2}$;	0
A_{12}	J_{k_1+q+1};	$q = 1, 2, \cdots, k_3 - k_1$;	q;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0
A_{131}, A_{133};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3$;	$k_3 - k_1 - q$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0
A_{132}, A_{134};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3$;	$k_3 - k_1 - q + 1$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	1
A_{141}	J_{k_2+q+1};	$q = 1, 2, \cdots, k_1$;	$\frac{(1)^{q+1} + (-1)^{q+1}}{2}$;	$\frac{(1)^{k_1+q} + (-1)^{k_1+q}}{2}$;	$\frac{(1)^{k_2-k_1+q+1} + (-1)^{k_2-k_1+q+1}}{2}$
A_{142}	J_{k_2+q+1};	$q = 1, 2, \cdots, k_1$;	$\frac{(1)^{q} + (-1)^{q}}{2}$;	$\frac{(1)^{k_1+q} + (-1)^{k_1+q}}{2}$;	$\frac{(1)^{k_2-k_1+q} + (-1)^{k_2-k_1+q}}{2}$
A_{151}	$J_{k_1+k_2+q+1}$;	$q = 1, 2, \cdots$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	$q - 1$;	$\frac{(1)^{k_2} + (-1)^{k_2}}{2}$
A_{152}	$J_{k_1+k_2+2}$;		$\frac{(1)^{k_1+1} + (-1)^{k_1+1}}{2}$;	0;	$\frac{(1)^{k_2+1} + (-1)^{k_2+1}}{2}$
	$J_{k_1+k_2+q+1}$;	$q = 2, 3, \cdots$;	Same zeros as in A_{151} for corresponding values of q.		

Cases	Polynomial	Range of Sub-Script	Zeros in $(-\infty, 0)$	Zeros in $(0, 1)$	Zeros in $(1, \infty)$
A_{211}, A_{221}	J_{k+1};	$0 \leq k \leq k_1$	0;	$\frac{(1)^k + (-1)^k}{2}$;	0
A_{212}, A_{222}	J_{k_1+q+1};	$q = 1, 2, \cdots, k_3 - k_1$;	q;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0
A_{213}	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3$;	$k_3 - k_1 - q$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0
A_{223}	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3$;	$k_3 - k_1 - q + 1$;	$\frac{(1)^{k_1+1} + (-1)^{k_1+1}}{2}$;	0
A_{214}, A_{224}	J_{k_2+q+1};	$q = 1, 2, \cdots k_1$;	0;	$\frac{(1)^{k_1-q} + (-1)^{k_1-q}}{2}$;	0
A_{215}	$J_{k_1+k_2+2}$;	$\begin{cases} J = \text{const}, > 0,\ k_1 \text{ odd.} \\ J = \text{const}, < 0,\ k_1 \text{ even.} \end{cases}$			
	$J_{k_1+k_2+q+2}$;	$q = 1, 2, \cdots$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	q;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$
A_{225}	$J_{k_1+k_2+2}$;	$\begin{cases} J = \text{const}, > 0,\ k_1 \text{ odd.} \\ \quad = \text{const}, < 0,\ k_1 \text{ even.} \end{cases}$			
	$J_{k_1+k_2+q+2}$;	$q = 1, 2, \cdots$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	$q - 1$;	$\frac{(1)^{k_1+1} + (-1)^{k_1+1}}{2}$

Cases	Polynomial	Range of Sub-Script	Zeros in			
			$(-\infty, 0)$	$x = 0$	$(0, 1)$	$(1, \infty)$
A_{311}, A_{321};	J_{k+1};	$0 \leq k < k_1$;	0;	0;	$\frac{(1)^k + (-1)^k}{2}$;	0
A_{312}, A_{322};	J_{k_1+q+1};	$q = 0, 1, \cdots, k_3 - k_1$;	q;	$k_1 + 1$;	0;	0
A_{313};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3$;	$k_3 - k_1 - q + 1$;	$k_1 + 1$;	0;	1
A_{323};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3$;	$k_3 - k_1 - q$;	$k_1 + 1$;	0;	0
A_{314};	J_{k_2+q+1};	$q = 1, 2, \cdots, k_1$;	0;	$k_1 + 1$;	0;	$\frac{(1)^q + (-1)^q}{2}$
A_{324};	J_{k_2+q+1};	$q = 1, 2, \cdots, k_1$;	0;	$k_1 + 1$;	0;	$\frac{(1)^{q+1} + (-1)^{q+1}}{2}$
A_{315};	$J_{k_1+k_2+q+1}$;	$q = 1, 2, 3, \cdots$;	0;	$k_1 + 1$;	$q - 1$;	$\frac{(1)^{k_1+1} + (-1)^{k_1+1}}{2}$
A_{325};	$J_{k_1+k_2+q+1}$;	$q = 1, 2, 3, \cdots$;	0;	$k_1 + 1$;	$q - 1$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$

Cases	Polynomial	Range of Sub-Script	Zeros in $(-\infty, 0)$	Zeros in $(0, 1)$	Zeros in $x = 1$	Zeros in $(1, \infty)$
A_{411}, A_{421};	J_{k+1};	$0 \leq k \leq k_1$;	0;	$\frac{(1)^k + (-1)^k}{2}$;	0;	0
A_{412}, A_{422};	J_{k_1+q+1};	$q = 1, 2, \cdots, k_3 - k_1$;	q;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0;	0
A_{413};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3 - 1$;	$k_3 - k_1 - q$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0;	1
A_{423};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3 - 1$;	$k_3 - k_1 - q$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	0;	0
A_{414}, A_{424};	J_{k_2+q+1};	$q = 0, 1, 2, \cdots, k_1$;	$\frac{(1)^{q+1} + (-1)^{q+1}}{2}$;	0;	$k_2 + 1$;	0
A_{415}, A_{425};	$J_{k_1+k_2+q+1}$;	$q = 1, 2, 3, \cdots$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$;	$q - 1$;	$k_2 + 1$;	0

A_5 : $\alpha < 0$, $\beta < 0$, $|\alpha| < |\beta|$, $\alpha = -k_1$ *integer*, $\beta = -k_2$ *integer*. Define k_1, k_2, k_3 as in A_1. In cases A_{51} and A_{52}, $\alpha + \beta$ is odd and even respectively.

Cases	Polynomial	Range of Sub-Script	Zeros in		
			$(-\infty, 0)$	$x = 0$	$(0, 1)$
A_{511}, A_{521};	J_{k+1};	$0 \leq k < k_1$;	0;	0;	$\frac{(1)^k + (-1)^k}{2}$;
A_{512}, A_{522};	J_{k_1+q+1};	$q = 0, 1, 2, \cdots, k_3 - k_1$;	q;	$k_1 + 1$;	0
A_{513};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3 - 1$;	$k_3 - k_1 - q$;	$k_1 + 1$;	0
A_{523};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_2 - k_3 - 1$;	$k_3 - k_1 - q + 1$;	$k_1 + 1$;	0
A_{514}, A_{524};	$J_{k_2+q+1} \equiv 0$;	$q = 0, 1, 2, \cdots k_1$.			
A_{515}, A_{525};	$J_{k_1+k_2+q+1} \equiv 0$;	$q = 1, 2, 3, \cdots$			

If assumptions are identical with those of A_5 except $|\alpha| = |\beta|$, then for $0 \leq k < k_1$, the results agree with A_{511} and $J_{k_1+q+1} \equiv 0$, $q = 0, 1, 2, \cdots$.

A_6 : $\alpha > 0$, $\beta < 0$, $|\alpha| > |\beta|$, β *not an integer*. Let k_1 be the largest integer in β.

Case	Polynomial	Range of Sub-Script	Zeros in	
			$(0, 1)$	$(1, \infty)$
A_{61}	J_{k+1}	$0 \leq k < k_1$	0	$\frac{(1)^k + (-1)^k}{2}$
A_{62}	J_{k_1+q+1}	$q = 1, 2, 3, \cdots$	q	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$

A_7 : *Same assumptions as in* A_6 *except* $\beta = -k_1$, *integer.*

Case	Polynomial	Range of Sub-Script	Zeros in		
			$(0, 1)$	$x = 1$	$(1, \infty)$
A_{71}	J_{k+1}	$0 \leq k \leq k_1 - 1$	0	0	$\frac{(1)^k + (-1)^k}{2}$
A_{72}	J_{k_1+q+1}	$q = 0, 1, 2, \cdots$	q	$k_1 + 1$	0

A_8 : $\alpha > 0$, $\beta < 0$, $|\alpha| = |\beta|$. $J_1 = \alpha$ and results for J_n, $n > 1$ are identical with those in A_7 and A_6 respectively according as β is or is not an integer.

A_9 : $\alpha > 0$, $\beta < 0$, $|\alpha| < |\beta|$; β, $\alpha + \beta$, *not integers.*

Let k_1 be the greatest integer in $\alpha + \beta$.
" k_2 " " " " " β.
" k_3 " " " " for which $\alpha + \beta + 2k < 0$.

Then $0 \leq k_3 \leq k_1 \leq k_2$.

Case	Polynomial	Range of Sub-Script	Zeros in $(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
A_{91};	J_{k+1};	$0 \leq k \leq k_3$;	$k + 1$;	0;	0
A_{921};	J_{k_3+q+1};	$q = 1, 2, \cdots, k_3$; k_1 even;	$k_3 - q + 1$;	0;	0
A_{922};	J_{k_3+q+1};	$q = 1, 2, \cdots, (k_3 + 1)$; k_1 odd;	$k_3 - q + 2$;	0;	1
A_{93};	J_{k_1+q+1};	$q = 1, 2, \cdots, (k_2 - k_1)$;	0;	0;	$\dfrac{(1)^{k_1+q} + (-1)^{k_1+q}}{2}$
A_{94};	J_{k_2+q+1};	$q = 1, 2, 3, \cdots$;	0;	q;	$\dfrac{(1)^{k_2} + (-1)^{k_2}}{2}$

A_{10} : *Same assumptions as in A_9 but now* $|\alpha| = |\beta|$. Then $k_1 = k_3 = 0$, $J_1 = \alpha$, and results for J_n, $n > 1$ are the same as in A_{93} and A_{94}.

A_{11} : *Same assumptions as in A_9 except $\beta = -k_2$, integer.*

Case	Polynomial	Range of Sub-Script	Zeros in $(-\infty, 0)$	$(0, 1)$	$x = 1$	$(1, \infty)$
$A_{11,1}$	Same as A_{91}					
$A_{11,2}$	Same as A_{92}					
$A_{11,3}$	Same as A_{93}					
$A_{11,4}$	J_{k_2+q+1};	$q = 1, 2, 3, \cdots$;	0;	q;	$k_2 + 1$;	0

A_{12} : $\alpha > 0$, $\beta < 0$, $|\alpha| < |\beta|$, *β not an integer.* $\alpha + \beta =$ *odd integer.* Define k_1, k_2, k_3 as in A_9.

A_{13} : *Same assumptions as in A_{12} except $\alpha + \beta =$ even integer.*

Cases	Polynomial	Range of Sub-Script	Zeros in $(-\infty, 0)$
$A_{12,1}$, $A_{13,1}$;	Same as A_{91}		
$A_{12,2}$;	$\begin{cases} J_{k_3+q+1}; \\ J_{2k_3+3} = \text{const.} > 0; \end{cases}$	$q = 1, 2, \cdots, k_3$;	$k_3 - q + 1$
$A_{13,2}$;	$\begin{cases} J_{k_3+q+1}; \\ J_{2k_3+3} = \text{const.} > 0; \end{cases}$	$q = 1, 2, \cdots, k_3 + 1$;	$k_3 - q + 2$
$A_{12,3}$, $A_{13,3}$;	Same as A_{93}		
$A_{12,4}$, $A_{13,4}$;	Same as A_{94}		

A_{14} : *Same assumptions as in* A_{12} , *except* $\beta = -k_2$ *integer.* Cases $A_{14,1}$, $A_{14,2}$ and $A_{14,3}$ have the same results as $A_{12,1}$, $A_{12,2}$, and $A_{12,3}$ respectively. $A_{14,4}$ has the same results as $A_{11,4}$.

A_{15} : *Same assumptions as* A_{13} *except* $\beta = -k_2$, *integer.* Cases $A_{15,1}$, $A_{15,2}$, and $A_{15,3}$ have the same results as $A_{13,1}$, $A_{13,2}$, and $A_{13,3}$ respectively. $A_{15,4}$ has the same results as $A_{11,4}$.

A_{16} : $\alpha = 0, \beta < 0, \beta$ — *not an integer.*

Let k_1 be the largest integer contained in β.

" k_3 be the largest integer for which $\beta + 2k < 0$.

Case	Polynomial	Range of Sub-Script	Zeros in $(-\infty, 0)$	$x = 0$	$(0, 1)$	$(1, \infty)$
$A_{16,1}$;	J_{k+1};	$0 \leq k \leq k_3$;	k;	1;	0;	0
$A_{16,2}$;	J_{k_3+q+1};	$q = 1, 2, \cdots, k_1 - k_3$;	$k_3 - q$;	1;	0;	0; k_1 even
			$k_3 - q + 1$;	1;	0;	1; k_1 odd
$A_{16,3}$;	J_{k_1+q+1};	$q = 1, 2, 3, \cdots$;	0;	1;	$q - 1$;	$\frac{(1)^{k_1} + (-1)^{k_1}}{2}$

A_{17} : $\alpha = 0, \beta = -k_1$ — *odd integer.* Define k_3 as in A_{16} .

A_{18} : $\alpha = 0, \beta = -k_1$ — *even integer.* Define k_3 as in A_{16} .

Cases	Polynomial	Range of Sub-Script	Zeros in $(-\infty, 0)$	$x = 0$	$(0, 1)$	$x = 1$
$A_{17,1}$, $A_{18,1}$;	Same as $A_{16,1}$					
$A_{17,2}$;	J_{k_3+q+1};	$q = 1, 2, \cdots, k_1 - k_3 - 1$;	$k_3 - q$;	1;	0;	0
$A_{18,2}$;	J_{k_3+q+1};	$q = 1, 2, \cdots, k_3 + 1$;	$k_3 - q + 1$;	1;	0;	0
$A_{17,3}$, $A_{18,3}$;	$J_{k_1+1} \equiv 0$					
	J_{k_1+q+1};	$q = 1, 2, 3, \cdots$;	0;	1;	$q - 1$;	$k_1 + 1$

A_{19} : $\alpha = 0, \beta = 0.$ $J_1 \equiv 0.$

J_{k+1} has $k - 1$ zeros in (0, 1), 1 zero at $x = 0$, 1 zero at $x = 1$, $k = 1, 2, 3, \cdots$.

From the definition of $J_n(x, \alpha, \beta)$ it is readily seen that $J_n(x, \alpha, \beta) \equiv (-1)^n J_n(1 - x, \beta, \alpha)$. Thus, a transformation of x to $1 - x$ interchanges α and β. The interval $(-\infty, 0)$ is transformed into $(1, \infty)$ and vice-versa. The points $x = 0$ and $x = 1$ are interchanged. Consequently, in all previous results we may interchange properly α and β.

In the foregoing results, the only real multiple zeros that can occur are at either $x = 0$ or $x = 1$. In the process of determining the degree of multiplicity of such zeros use was made of Theorem I_2 .

Points of Inflexion. By taking (4), setting $k = n$, and replacing N' and D''

by their values for Jacobi polynomials, we get: $P''_{n+1}(n, x) = (n + 1)(n)[\beta + \alpha + n][\beta + \alpha + n + 1] P_{n-1}(n, x)$. From definitions of $P_n(k, x)$ and $J_n(x, \alpha, \beta)$ we easily verify that,

$$P_n(n \pm q, x) \equiv J_n(x, \alpha \pm q + 1, \beta \pm q + 1), \text{ whence,}$$

$$J''_n(x, \alpha, \beta) = (n + 1)(n)[\beta + \alpha + n][\beta + \alpha + n + 1] J_{n-1}(x, \alpha + 2, \beta + 2).$$

We conclude that if neither $\alpha + \beta + n$ nor $\alpha + \beta + n + 1$ vanishes, the points of inflexion of $J_{n+1}(x, \alpha, \beta)$ are at the zeros of odd order of $J_{n-1}(x, \alpha + 2, \beta + 2)$.

The Degree of $J_n(x, \alpha, \beta)$. In analyzing the results of cases A_1 to A_{19} inclusive, it is noted that in some cases the number of real zeros of J_n is less than n. The question naturally arises whether the degree of J_n is n or less, for then we can determine the number of its imaginary zeros. The explicit expression of $J_n(x, \alpha, \beta)$ is known from which the degree of J_n can be found for various α and β. However, the degree of J_n can be found from (4).

Since $J_{n+1}(x, \alpha, \beta) \equiv P_{n+1}(n, x)$, let us replace k by n in (4) and at the same time replace N' and D'' by their values for Jacobi Polynomials. Thus, we get:

$$J^{(q)}_{n+1}(x, \alpha, \beta) = \prod_{i=0}^{q-1} (n + 1 - i)[-\beta - \alpha - n - i] P_{n-q+1}(n, x), \tag{5}$$

$$n = 0, 1, 2, \cdots; q = 0, 1, \cdots, (n + 1).$$

We may establish the following results.

C_1) If $\alpha + \beta$ is not an integer, the degree of $J_{n+1}(x, \alpha, \beta)$ is $n + 1$, $n = 0, 1, 2, \cdots$.

In fact, in order for $J^{(q)}_{n+1}$ to vanish, we see from (5) that either some factor $-\beta - \alpha - n - i$ vanishes or $P_{n-q+1}(n, x)$ vanishes identically. We first show that the latter is not possible. Now $P_1(n, x) \equiv N + nD' \equiv (-\beta - \alpha - 2n)x + \alpha + n \not\equiv 0$ since $\beta + \alpha$ is not an integer. Consequently, if $P_\mu(n, x) \equiv 0$, $\mu > 0$, $\mu \leq n + 1$ there will be a first value of μ, ($\mu = \nu$), for which $P_\nu(n, x) \equiv 0$ but $P_{\nu-1}(n, x) \not\equiv 0$. By virtue of Theorem I_7 this means that either $N + (n - p)D' \equiv [-\beta - \alpha - 2(n - p)]x + \alpha + n - p \equiv 0$, $p \leq \nu$, or else there exist two values of p, (p_1, p_2), such that $[-\beta - \alpha - 2(n - p_1)]x + \alpha + n - p_1$ and $[-\beta - \alpha - 2(n - p_2)]x + \alpha + n - p_2$ are divisible by x and $1 - x$ respectively, $p_1, p_2 \leq \nu - 1$, $p_1 \neq p_2$. Since, however, $\alpha + \beta$ is not an integer we see that, $[-\beta - \alpha - 2(n - p)]x + \alpha + n - p \not\equiv 0$, n and p being integers. This eliminates the first possibility that $P_\mu(n, x) \equiv 0$, $\mu \leq n + 1$. Again, if, $[-\beta - \alpha - 2(n - p_1)]x + \alpha + n - p_1$ is divisible by x, we have $\alpha + n - p_1 = 0$ or α an integer. For $(\alpha + n - p_2) - [\beta + \alpha + 2(n - p_2)]x \equiv (\alpha + n - p_2)\left[1 - \frac{(\alpha + n - p_2) + (\beta + n - p_2)}{(\alpha + n - p_2)} x\right]$ to be divisible by $1 - x$ requires $\beta + n - p_2 = 0$ or β, an integer. α and β are therefore both integers contrary to hypothesis. Thus, in (5), no polynomial $P_{n-q+1}(k, x) \equiv 0$ and $J_{n+1}(x, \alpha, \beta) \not\equiv 0$. Replacing q by $n + 1$ in (5) leads to,

$$J_{n+1}^{(n+1)}(x, \alpha, \beta) = \prod_{i=0}^{n} (n + 1 - i)[-\beta - \alpha - n - i] P_0(n, x), \tag{6}$$
$$n = 0, 1, 2, \cdots.$$

Thus $J_{n+1}^{(n+1)} \not\equiv 0$, (since $P_0(n, x) = 1$ and no factor $-\beta - \alpha - n - i$ can vanish) and the degree of J_{n+1} is precisely $n + 1$. From similar reasoning we prove:

C_2) If $\alpha + \beta > 0$ the degree of J_{n+1} is $n + 1$, $n = 0, 1, 2, \cdots$.

C_3) If $\alpha + \beta = 0$, then (I) $J_1 = \alpha$ and (II) J_{n+1} is of degree $n + 1$, $n = 1, 2, 3, \cdots$.

C_4) If $\alpha + \beta = -M$ – integer, $M > 0$, β, α not integers, then,
(I) For $n < M$, the degree of J_{n+1} is min. $(n + 1, M - n)$.
(II) $n = M$, $J_{n+1} \equiv$ const.
(III) $n > M$, the degree of J_{n+1} is $n + 1$.

C_5) If $\alpha + \beta = -M$ – integer, $M > 0$, α, β integers, $\alpha > 0$, $\beta < 0$, then,
(I) For $n < M$, the degree of J_{n+1} is min. $(n + 1, M - n)$.
(II) $n = M$, $J_{n+1} \equiv$ const.
(III) $n > M$, the degree of J_{n+1} is $n + 1$.

C_6) If $\alpha + \beta = -M$ – integer, $M > 0$, $\alpha = -k_1$-integer, $\beta = -k_2$-integer, $k_1 < k_2$ then,
(I) For $n < k_2$, J_{n+1} is of degree $n + 1$.
(II) $n \geq k_2$, $J_{n+1} \equiv 0$.

C_7) If $\alpha + \beta = -M$ – integer, $M > 0$, $\alpha = \beta = -k_1$-integer, then,
(I) For $n < k_1$, J_{n+1} is of degree $n + 1$,
(II) $n \geq k_1$, $J_{n+1} \equiv 0$.

The Laguerre Polynomials. These are defined as follows:

$$L_n \equiv L_n(x, \alpha) = x^{1-\alpha} e^x \frac{d^n}{dx^n} [e^{-x} x^{n+\alpha-1}], \qquad n = 0, 1, 2, \cdots;$$

α – real. We see that L_n is a special case of $P_n(k, x)$ with $N \equiv -x + \alpha$, $D \equiv x$, $n = k + 1$. It follows that $\theta = -1$, $\mu = \alpha + k - n$, $\alpha_{1k1} = \alpha + k$, and $P_1'(k, x) = 1$. These can be used in determining the location of the real zeros of L_n, as was done for J_n. The discussion here is somewhat simplified since L_n has but one parameter, α, and the x-axis is divided by the zeros of $D(x)$ into two segments only, namely, $(-\infty, 0)$ and $(0, \infty)$.

The following results are easily obtained.

B_1 : $\alpha > 0$, $L_n(x, \alpha)$ has n distinct zeros in $(0, \infty)$, $n = 1, 2, 3, \cdots$. This result is well known.

B_2 : $\alpha = 0$. $L_{n+1}(x, \alpha)$ has n distinct zeros in $(0, \infty)$ and a simple zero at $x = 0$, $n = 0, 1, 2, \cdots$.

B_3 : $\alpha < 0$, *α, not an integer.* Let k_1 be the largest integer contained in α.

(I) $L_{k+1}(x, \alpha)$ has $\dfrac{(1)^k + (-1)^k}{2}$ zeros in $(-\infty, 0)$, $0 \leq k \leq k_1$,

(II) $L_{k_1+q+1}(x, \alpha)$ has q distinct zeros in $(0, \infty)$ and $\dfrac{(1)^{k_1} + (-1)^{k_1}}{2}$ zeros in $(-\infty, 0)$, $q = 0, 1, 2, \cdots$.

B_4 : $\alpha < 0$, $\alpha = -k_1$ — *integer.*

(I) $L_{k+1}(x, \alpha)$ has $\dfrac{(1)^k + (-1)^k}{2}$ zeros in $(-\infty, 0)$, $0 \leq k \leq k_1$.

(II) $L_{k_1+q+1}(x, \alpha)$ has q distinct zeros in $(0, \infty)$ and a zero of order $k_1 + 1$ at $x = 0$, $q = 0, 1, 2, \cdots$.

The Degree of $L_n(x, \alpha)$. We show first that here $P\mu(n, x) \not\equiv 0$, $\mu = 1, 2, \cdots n + 1$. By definition, $P_1(n, x) \equiv N + nD' \equiv -x + \alpha + n \not\equiv 0$. Let us rewrite (2) for our present situation thus:

(2°) $P'_\mu(n, x) = -\mu P_{\mu-1}(n, x)$. If, now, $P_\mu(n, x) \equiv 0$, then from (2°) it follows that $P_{\mu-1}(n, x) \equiv 0$. Continuing this reasoning, we finally arrive at a contradiction, namely, $P_1(n, x) \equiv 0$. If in (4) we set $q = n + 1$ and replace N' and D'' by their values we get:

$$L_{n+1}^{(n+1)}(x, \alpha) = (-1)^{n+1}(n + 1)! \qquad P_0(n, x) = (-1)^{n+1}(n + 1)!$$

Hence, L_{n+1} is of degree $n + 1$. Note that this holds regardless of the value of α contrary to what was found for Jacobi Polynomials.

Points of Inflexion. By a procedure analogous to that used for Jacobi Polynomials we can show that the points of inflexion of $L_{n+1}(x, \alpha)$ are located at the zeros of odd order of $L_{n-1}(x, \alpha + 2)$.

The Polynomials $P_n(0, x)$. If we set $k = 0$ in (1), (2), and (3) we obtain the following relationships for $P_n(0, x)$[5] $\equiv P_n(x) \equiv P_n$.

(7) $P_{n+1}(x) = [N - nD']\, P_n(x) + DP'_n(x)$.

(8) $P'_{n+1}(x) = (n + 1)\left[N' - \dfrac{n}{2} D''\right] P_n(x)$.

(9) $P_{n+1}(x) = [N - nD']\, P_n(x) + n\left(N' - \dfrac{n-1}{2} D''\right) DP_{n-1}(x)$.

Theorems I_1 to I_{10} inclusive, with $k = 0$, hold for $P_n(x)$. In addition, the following theorems hold for P_n.

Theorem H_1. *Suppose* N *linear and* $D(x) > 0$ *for all* x. *Furthermore, let* $N' - \dfrac{m}{2} D'' < 0$, $m = 1, 2, 3, \cdots$. *Then* P_n *has* n *real, distinct zeros which separate the zeros of* P_{n+1}.

Proof: Denote the zeros of P_n by $\alpha_{n,i}$, $i = 1, 2, \cdots n$, $\alpha_{n,i} < \alpha_{n,i+1}$. Suppose $N' > 0$. N being linear has a single zero α_{11}. Furthermore, since $P_1 \equiv N_1$, then $P_1 < 0$ for $x < \alpha_{11}$ and > 0 for $x > \alpha_{11}$. We pass now to P_2. From (7), we see that $P_2(\alpha_{11}) > 0$, (since $D > 0$ and $P'_1 > 0$). Also (8) shows $P'_2(x) > 0$

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[5] E. H. Hildebrandt, loc. cit. pp. 399.

for $x < \alpha_{11}$ and < 0 for $x > \alpha_{11}$. This follows from what was noted concerning the sign of P_1 for $x > \alpha_{11}$ and $x < \alpha_{11}$, together with the hypothesis that $N' - \frac{m}{2} D'' < 0$. Thus, there exists a zero of P_2 in $(-\infty, \alpha_{11})$ and a zero in (α_{11}, ∞) and our theorem holds for $n = 1$. Assume that the theorem is true for $n = h$. The sequence $P_h'(\alpha_{h,i})$, $i = 1, 2, \cdots h$, is alternately positive and negative. Since, from (8), the only extrema of P_{h+1} are at $\alpha_{h,i}$, $i = 1, 2, \cdots h$, we conclude that there are $h - 1$ zeros of P_{h+1} separating the $\alpha_{h,i}$, $i = 1, 2, \cdots h$. Since $P_h'(\alpha_{h,1}) > 0$ we conclude that $P_h < 0$ for $x < \alpha_{h,1}$. This fact, combined with (8), shows $P_{h+1}'(x) > 0$ for $x < \alpha_{h,1}$. $P_{h+1}(\alpha_{h,1})$ being positive, it follows that there exists a zero of P_{h+1} in $(-\infty, \alpha_{h,1})$. Similar reasoning establishes the existence of a zero of P_{h+1} in $(\alpha_{h,h}, \infty)$. Our theorem is thus established for $N' > 0$. The case $N' < 0$ can be similarly treated.

Theorem H_2 : *If* $D(x) > 0$ *for all* x, $D'' < 0$, $N' - \frac{m}{2} D'' < 0$, $N' = 0$, $N \not\equiv 0$, *then* P_n, $n = 2, 3, \cdots$, *has* $n - 1$ *real, distinct zeros which are separated by the zeros of* P_{n-1}.

Proof: Since $P_1 \equiv N = \text{const.}$, we see from (7) that P_2 is linear. The reasoning of Theorem H_1 applies where we now start with P_2.

Theorem H_3 : *If* $D(x) > 0$ *for all* x, *except* $x = \beta$, *where* D *has a double zero and if* $N' \not\equiv 0$, $N' - \frac{n}{2} D'' < 0$, $n = 1, 2, 3, \cdots$, *then* P_n *has* n *real, distinct zeros which separate those of* P_{n+1}.

Proof: Theorem I_1 with $k = 0$ assures us that P_n and D have no zeros in common. The proof now follows the line of reasoning of Theorem H_1.

Theorem H_4 : *If* $D(x) > 0$ *for all* x *except* $x = \beta$ *where* D *has a double zero and if* $N' = 0$, $N \not\equiv 0$, $N' - \frac{m}{2} D'' < 0$, $m = 1, 2, 3, \cdots$, *then* P_n *has* $n - 1$ *real, distinct zeros which separate those of* P_{n+1}, $n = 1, 2, 3, \cdots$. *This theorem follows from* H_3 *as did* H_2 *from* H_1.

Points of Inflexion. Setting $k = 0$ in (4) leads to,

$$P_{n+1}'' = (n + 1)\,(n) \left[N' - \frac{n}{2} D''\right]\left[N' - \frac{n-1}{2} D''\right] P_{n-1}.$$

This shows, under the assumptions of Theorems H_1 to H_4 inclusive, that the points of inflexion of P_{n+1} are at the zeros of P_{n-1}.

Hermite Polynomials. Theorem H_1 and statement immediately above concerning points of inflexion apply directly to Hermite Polynomials where $N \equiv -x$ and $D \equiv \sigma^2$.

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THE SIMULTANEOUS COMPUTATION OF GROUPS OF REGRESSION EQUATIONS AND ASSOCIATED MULTIPLE CORRELATION COEFFICIENTS

By Paul S. Dwyer

1. **Introduction.** The need sometimes arises for the prediction of a number of different variables from a given group of so-called fundamental variables. In the work of college prediction, for example, one might desire regression equations predicting certain measures of college achievement (e.g., first semester average, first semester English grade, first semester mathematics grade, number of hours of A received during first semester, etc.) on the basis of a number of other factors (e.g., high school record, score on American Council on Education Psychological Examination, score on some standard English achievement test, score on some standard mathematics achievement test, etc.). It is the purpose of this paper to show how the regression coefficients and the associated multiple correlation coefficients can be obtained simultaneously. The essence of the method is a simple device by which one solution of general normal equations may be made to serve for all cases.

2. **The normal equations.** Let $x_1, x_2, x_3, \cdots x_n$, be the so-called fundamental variables and let x_k be the predicted variable. The normal equations are computed by standard methods which result in one of the three types.
Type I. Normal equations for determining $b_0, b_1, b_2, b_3, \cdots, b_n$.

$$\begin{array}{l}
b_0 n + b_1\Sigma x_1 + b_2\Sigma x_2 + b_3\Sigma x_3 + \cdots\cdots + b_n\Sigma x_n - \Sigma x_k = 0 \\
b_0\Sigma x_1 + b_1\Sigma x_1^2 + b_2\Sigma x_1x_2 + b_3\Sigma x_1x_3 + \cdots\cdots + b_n\Sigma x_1x_n - \Sigma x_1x_k = 0 \\
b_0\Sigma x_2 + b_1\Sigma x_1x_2 + b_2\Sigma x_2^2 + b_3\Sigma x_2x_3 + \cdots\cdots + b_n\Sigma x_2x_n - \Sigma x_2x_k = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
b_0\Sigma x_n + b_1\Sigma x_nx_1 + b_2\Sigma x_nx_2 + b_3\Sigma x_nx_3 + \cdots\cdots + b_n\Sigma x_n^2 - \Sigma x_nx_k = 0
\end{array}$$

Type II. Normal equations for determining $b_1, b_2, b_3, \cdots, b_n$.

$\bar{x}_i = x_i - M_{x_i}$

$$\begin{array}{l}
b_1\Sigma\bar{x}_1^2 + b_2\Sigma\bar{x}_1\bar{x}_2 + b_3\Sigma\bar{x}_1\bar{x}_3 + \cdots\cdots + b_n\Sigma\bar{x}_1\bar{x}_n - \Sigma\bar{x}_1\bar{x}_k = 0 \\
b_1\Sigma\bar{x}_2\bar{x}_1 + b_2\Sigma\bar{x}_2^2 + b_3\Sigma\bar{x}_2\bar{x}_3 + \cdots\cdots + b_n\Sigma\bar{x}_2\bar{x}_n - \Sigma\bar{x}_2\bar{x}_k = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
b_1\Sigma\bar{x}_n\bar{x}_1 + b_2\Sigma\bar{x}_n\bar{x}_2 + b_3\Sigma\bar{x}_n\bar{x}_3 + \cdots\cdots + b_n\Sigma\bar{x}_n^2 - \Sigma\bar{x}_n\bar{x}_k = 0
\end{array}$$

Type III. Normal equations for determining $\beta_1, \beta_2, \beta_3, \cdots, \beta_n$.

$$\beta_1 + r_{12}\beta_2 + r_{13}\beta_3 + \cdots\cdots + r_{1n}\beta_n - r_{1k} = 0$$
$$r_{21}\beta_1 + \beta_2 + r_{23}\beta_3 + \cdots\cdots + r_{2n}\beta_n - r_{2k} = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$r_{n1}\beta_1 + r_{n2}\beta_2 + r_{n3}\beta_3 + \cdots\cdots + r_{nn}\beta_n - r_{nk} = 0$$

The three types are special cases of the general

$$d_{11}y_1 + d_{12}y_2 + d_{13}y_3 + \cdots + d_{1j}y_j + \cdots + d_{1n}y_n - d_{1k} = 0$$
$$d_{21}y_1 + d_{22}y_2 + d_{23}y_3 + \cdots + d_{2j}y_j + \cdots + d_{2n}y_n - d_{2k} = 0$$
$$d_{31}y_1 + d_{32}y_2 + d_{33}y_3 + \cdots + d_{3j}y_j + \cdots + d_{3n}y_n - d_{3k} = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$d_{i1}y_1 + d_{i2}y_2 + d_{i3}y_3 + \cdots + d_{ij}y_j + \cdots + d_{in}y_n - d_{ik} = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$d_{n1}y_1 + d_{n2}y_2 + d_{n3}y_3 + \cdots + d_{nj}y_j + \cdots + d_{nn}y_n - d_{nk} = 0$$

where y_j are the regression coefficients and $d_{ij} = d_{ji}$.

The methods described in this paper are applicable to the general case and hence to each of the three particular types.

In examining the normal equations, it is noticed that the first n terms of each equation are completely determined by the n fundamental variables. The equations, aside from the last terms, are identical no matter what variable is predicted. It is only necessary to devise a technique for separating the contributions of the d_{ik} terms.

3. Solution by determinants. One method utilizes determinants. The value y_j is expressed in terms of a determinant involving a column with entries $d_{1k}, d_{2k}, d_{3k}, \cdots, d_{nk}$. The determinant is expanded in terms of this column.

Specifically, let D be the determinant of the coefficients of the y_j and let D_{ij} be the cofactor of any element d_{ij} of D. Then

$$D = \sum_{i=1}^{n} D_{ij}\, d_{ij}$$

and

$$y_1 = \frac{1}{D}(D_{11}\,d_{1k} + D_{21}\,d_{2k} + D_{31}\,d_{3k} + \cdots + D_{j1}\,d_{jk} + \cdots + D_{n1}\,d_{nk}.)$$

$$y_2 = \frac{1}{D}(D_{12}\,d_{1k} + D_{22}\,d_{2k} + D_{32}\,d_{3k} + \cdots + D_{j2}\,d_{jk} + \cdots + D_{n2}\,d_{nk}.)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_i = \frac{1}{D}(D_{1i}\,d_{1k} + D_{2i}\,d_{2k} + D_{3i}\,d_{3k} + \cdots + D_{ji}\,d_{jk} + \cdots + D_{ni}\,d_{nk}.)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_n = \frac{1}{D}(D_{1n}\,d_{1k} + D_{2n}\,d_{2k} + D_{3n}\,d_{3k} + \cdots + D_{jn}\,d_{jk} + \cdots + D_{nn}\,d_{nk}.)$$

It is only necessary to compute $\frac{D_{ji}}{D}$ to find the coefficient of d_{jk} in the expansion of y_i.

An illustration is given. The normal equations are

$$\beta_1 + .3300\,\beta_2 + .2100\,\beta_3 - r_{1k} = 0$$

$$.3300\,\beta_1 + \beta_2 - .4800\,\beta_3 - r_{2k} = 0$$

$$.2100\,\beta_1 - .4800\,\beta_2 + \beta_3 - r_{3k} = 0$$

from which at once

$$\beta_1 = \frac{1}{D}(.7696\,r_{1k} - .4308\,r_{2k} - .3684\,r_{3k})$$

$$\beta_2 = \frac{1}{D}(-.4308\,r_{1k} + .9559\,r_{2k} + .5493\,r_{3k})$$

$$\beta_3 = \frac{1}{D}(-.3684\,r_{1k} + .5493\,r_{2k} + .8911\,r_{3k})$$

and also

$$\begin{aligned} D = .550072 &= (1.00)(.7696) + (.33)(-.4308) + (.21)(-.3684) \\ &= (.33)(-.4308) + (\ 1.00)(.9559) + (-.48)(.5493) \\ &= (.21)(-.3684) + (-.48)(.5493) + (\ 1.00)(.8911) \end{aligned}$$

so that

$$\beta_1 = \ \ 1.3991\,r_{1k} - \ \ .7832\,r_{2k} - \ \ .6697\,r_{3k}\,.$$

$$\beta_2 = -.7832\,r_{1k} + 1.7378\,r_{2k} + \ \ .9986\,r_{3k}\,.$$

$$\beta_3 = -.6697\,r_{1k} + \ \ .9986\,r_{2k} + 1.6200\,r_{3k}\,.$$

It is only necessary to insert any given values r_{1k}, r_{2k}, r_{3k}, to obtain the coefficients of any specific regression equation.

4. **Solutions without determinants.** Theoretically the solution by determinants is excellent but as the number of variables increases the work of computing the n^2 cofactors $\left[\text{or the } \frac{n(n+1)}{2} \textit{ different} \text{ cofactors}\right]$ becomes enormous. We desire a technique for separating the contributions of the last terms when determinants are not used. This can be accomplished by using a separate column for each d_{ik}. Before algebraic manipulation, the value d_{ik} is factored from the column and, after manipulative solution is complete, the multiplication by d_{ik} is carried out.

As an example consider the normal equations

$$\beta_1 + r_{12}\beta_2 - r_{1k} = 0$$

$$r_1\beta_{21} + \beta_2 - r_{2k} = 0$$

where $r_{12} = r_{21} = .3300$. Then the normal equations may be represented by rows (1) and (2) of Table I.

TABLE I

Row	Operation	β_1	β_2	r_{1k}	r_{2k}
(1)		1.0000	.3300	−1.0000	
(2)		.3300	1.0000		−1.0000
(3)	− .3300 times (2)	− .1089	− .3300		.3300
(4)	(1) + (3)	.8911		−1.0000	.3300
(5)	− (4) divided by .8911	−1.0000		1.1222	− .3703
(6)	− .3300 times (5)	.3300		− .3703	.1222
(7)	− (2) + (6)		−1.0000	− .3703	1.1222

The four decimal place solution, whose steps are indicated by (3) (4) (5) (6)(7), is from (5) and (7)

$$\beta_1 = 1.1222\, r_{1k} - .3703\, r_{2k}$$

$$\beta_2 = -.3703\, r_{1k} + 1.1222\, r_{2k}$$

This device may be combined with most of the standard methods of solving normal equations.

5. **Combination with Doolittle method.** Especially to be recommended is a combination of this device with the Doolittle method which is recognized as a most efficient method of solving normal equations in from five to ten variables [1] [2]. One of the advantages of the Doolittle method is that related multiple regression coefficients may be obtained from the same forward solution, though additional back solutions are necessary [3].

The problem which led to the development of this technique was the simultaneous prediction of scores on various occupations covered by the Strong Vocational Interest Blank from the scores on a few fundamental occupations. A multiple factor analysis revealed that five basic factors account for most of the scores. Five occupational scores, serving as approximations to the five basic factors, were used as the fundamental variables and the other scores were predicted from them.

As an illustration of this prediction technique combined with the Doolittle method, I have selected three test scores as fundamental since the solution based on them shows all the steps of the Doolittle method and is shorter than the five

variable problem. Actually, solution by determinants (section 3) is advised for problems involving three variables. The steps of the Doolittle solution are presented in Table II. The results should be compared with those of the determinant solution of section 3.

The first column indicates the row and the second the description of the algebraic operation. The next three columns are the standard columns of a Doolittle presentation with the conventional elimination of the lower left entries. The next three columns carry through the Doolittle method with the values r_{1k}, r_{2k}, r_{3k} kept in separate columns. The last column is an adaptation of the conventional summary check column of the Doolittle solution.

TABLE II

Generalized Doolittle Presentation

Row	Operation	β_1	β_2	β_3	r_{1k}	r_{2k}	r_{3k}	S
(1)		1.0000	.3300	.2100	−1.0000			.5400
(2)		.3300	1.0000	−.4800		−1.0000		−.1500
(3)		.2100	−.4800	1.0000			−1.0000	−.2700
(4)	Repeat (1)	1.0000	.3300	.2100	−1.0000			.5400
(5)	Negative of (4)	−1.0000	−.3300	−.2100	1.0000			−.5400
(6)	Repeat (2)		1.0000	−.4800		−1.0000		−.1500
(7)	−.3300 times (4)		−.1089	−.0693	.3300			−.1782
(8)	(6) + (7)		.8911	−.5493	.3300	−1.0000		−.3282
(9)	−(8) divided by .8911		−1.0000	.6164	−.3703	1.1222		.3683
(10)	Repeat (3)			1.0000			−1.0000	−.2700
(11)	−.2100 times (4)			−.0441	.2100			−.1134
(12)	.6164 times (8)			−.3386	.2034	−.6164		−.2023
(13)	(10) + (11) + (12)			.6173	.4134	−.6164	−1.0000	−.5857
(14)	−(13) divided by .6173			−1.0000	−.6697	.9985	1.6200	.9488
(15)	.6164 times (14)			−.6164	−.4128	.6155	*.9986*	.5848
(16)	(9) + (15)		−1.0000		−.7831	1.7377	*.9986*	.9531
(17)	−.2100 times (14)			.2100	.1406	−*.2097*	−*.3402*	−.1992
(18)	−.3000 times (16)		.3300		.2584	−*.5734*	−*.3295*	−.3145
(19)	(5) + (17) + (18)	−1.0000			1.3990	−*.7831*	−*.6697*	−1.0537

The general solution is read from rows (19) (16) (14) and is

$$\beta_1 = 1.3990\, r_{1k} - .7831\, r_{2k} - .6697\, r_{3k}\,.$$

$$\beta_2 = -.7831\, r_{1k} + 1.7377\, r_{2k} + .9986\, r_{3k}\,.$$

$$\beta_3 = -.6697\, r_{1k} + .9985\, r_{2k} + 1.6200\, r_{3k}\,.$$

which agrees, aside from the last place, with the result of the solution by determinants.

It is wise to check in the original equations (1), (2), (3) as soon as any β_i is found. Row (14), for example, should be checked by showing

$$(-.6697)(1.00) + (.9985)(\ .33) + (1.6200)(\ .21) = \ \ .0000$$

$$(-.6697)(\ .33) + (.9985)(\ 1.00) + (1.6200)(-.48) = -.0001$$

$$(-.6697)(\ .21) + (.9985)(-.48) + (1.6200)(\ 1.00) = \ 1.0001$$

The same should be done with row (16) as soon as it is computed. Row (19) should be treated similarly.

6. **Many regression equations.** If large numbers of regression equations are to be generated (the Strong Vocational Interest Study had 29 dependent variables), the following technique is suggested. Make a table with columns r_{1k}, r_{2k}, etc. and use the rows to indicate the different values of k. On another slip of paper insert the general values β_1, β_2, β_3, $\cdots$ β_n in successive rows so that a folding of the paper will bring any general β expansion in conjunction with the r's of any test, k. The scheme is illustrated in Table III.

TABLE III

No.	Occupation	r_{1k}	r_{2k}	r_{3k}		β_{1k}	β_{2k}	β_{3k}		r
1	Teacher	1.00	.33	.21		1.00	.00	.00		1.00
2	Physicist	.33	1.00	−.48		.00	1.00	.00		1.00
3	Office Worker	.21	−.48	1.00		.00	.00	1.00		1.00
4	Doctor	.17	.79	−.52		−.03	.72	−.17		.81
5	Lawyer	−.02	.16	−.59		.24	−.30	−.78		.64
6	Engineer	.16	.78	−.02		−.37	1.21	.64		.93
	β_1	1.3990	−.7831	−.6697		1.0000				
	β_2	−.7831	1.7377	.9986			1.0000			
	β_3	−.6697	.9985	1.6200				1.0000		
10	Mathematician etc.	.46	.96	−.49		.19	.82	−.14		.97

Thus, for the occupation of Engineer,

$$\beta_1 = \ 1.3990\ (.16) + (-.7831)(.78) + (-.6697)(-.02) = -.37$$

$$\beta_2 = -.7831\ (.16) + (\ 1.7377)(.78) + (\ \ .9996)(-.02) = \ 1.21$$

$$\beta_3 = -.6697\ (.16) + (\ \ .9985)(.78) + (\ 1.6200)(-.02) = \ \ .64$$

The value of the multiple correlation coefficient is then computed from the formula

$$r_{k.123--n} = \sqrt{\beta_{1k} r_{1k} + \beta_{2k} r_{2k} + \ldots + \beta_{nk} r_{nk}}$$

In the illustration above

$$r_{k.123} = \sqrt{(-.37)(.16) + (1.21)(.78) + (.64)(-.02)}$$
$$= .93$$

7. **Regression equations by deletion.** The method of getting related regression coefficients and correlation coefficients, described by Kurtz [3], is also applicable. Again, a problem involving more than three variables is needed to show the real value of the scheme but the technique may be illustrated in the three variable case. We wish to find, from the forward solution of Table II, the regression equation and the multiple correlation coefficient when the first two fundamental variables only are used. We delete all columns involving test 3 and complete the back solution as indicated in Table IV, which may be viewed as a substitute for the last ten rows of Table II.

TABLE IV
(See Table II)

Row	Operation	β_1	β_2	β_3	r_{1k}	r_{2k}	r_{3k}	S
(20)	Repeat (9)		-1.0000		$-.3703$	1.1222		
(21)	$-.3300$ times (20)		$.3300$		$.1222$	$-.3703$		
(22)	(5) + (21)	-1.0000			1.1222	$-.3703$		

The results are

$$\beta_1 = 1.1222\, r_{1k} - .3703\, r_{2k}\,.$$

$$\beta_2 = -.3703\, r_{1k} + 1.1222\, r_{2k}\,.$$

and these agree with the results of section 4.

8. **The simplified back solution.** In every case in which the β's have been given in terms of r's the matrix of the coefficients is symmetric (sections 3, 4, 5, 7). One wonders if this symmetry is generally true and if it holds for normal equations of Type I or Type II.

Determinants are much more useful in establishing general properties, such as the one under discussion, than they are in computing the values of regression coefficients in the case of a problem involving many variables. We return to the determinant notation of section 3.

In each of the three types, and hence in the general case $d_{ij} = d_{ji}$ so that D is a symmetric determinant, $D_{ij} = D_{ji}$ and $\frac{D_{ij}}{D} = \frac{D_{ji}}{D}$. Hence the matrix of the coefficients of the solution is symmetric.

This result may be used (1) to check the expanded results or (2) to eliminate some of the work of the back solution. The n coefficients must be recorded for β_n after which the column indicated by r_{nk} may be dropped. The first $n - 1$ coefficients must be computed for β_{n-1} after which the column indicated by $r_{n-1,k}$ may be dropped, etc. The italicized entries in Table II are the ones which are eliminated in this way. The remaining coefficients are sufficient to completely determine the symmetric matrix.

The summary right hand check column can not be readily used in the simplified back solution but it is hardly to be recommended anyway. Kurtz [3] argues against it on the ground that it is not necessary. The essential check is to see that each β solution satisfies all of the original equations.

9. **Conclusion.** This paper provides a technique for the computation of general regression equations and shows how the technique may be combined with the Doolittle method in providing a practical means of mass prediction.

UNIVERSITY OF MICHIGAN.

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[1] TOLLEY, H. R. AND EZEKIAL, MORDECAI. *The Doolittle method for solving multiple correlation equations versus the Kelley-Salisbury Iteration method.* Journal of American Statistical Association, 1927(22), pp. 497–500.

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[3] KURTZ, A. K. *The use of the Doolittle method in obtaining related multiple correlation coefficients.* Psychometrika, Vol. I, no. 1, March 1936, pp. 45–51.

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GRIFFIN, H. D. *On partial correlation versus partial regression for obtaining the multiple regression equations.* Jour. of Ed. Psy. 1939(22), pp. 35–44.

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CONSTITUTION

Article I

NAME AND PURPOSE

1. This organization shall be known as the Institute of Mathematical Statistics.

2. Its object shall be to promote the interests of mathematical statistics.

Article II

MEMBERSHIP

1. The membership of the Institute shall consist of Members, Fellows, Honorary Members, and Sustaining Members.

2. Fellows shall be the only voting members of the Institute.

Article III

OFFICERS, BOARD OF DIRECTORS, COMMITTEE ON MEMBERSHIP, AND COMMITTEE ON PUBLICATIONS

1. The Officers of the Institute shall be a President, two Vice-Presidents, and a Secretary-Treasurer, elected for a term of one year by a majority ballot at the annual meeting of the Institute. Voting may be in person or by mail.

(a) Exception. The first group of Officers shall be elected by a majority vote of the individuals present at the organization meeting, and shall serve until December 31, 1936.

2. The Board of Directors of the Institute shall consist of the Officers and the previous President.

3. The Institute shall have a Committee on Membership composed of three Fellows. At their first meeting subsequent to the adoption of this Constitution, the Board of Directors shall elect three members as Fellows to serve as the Committee on Membership, one member of the Committee for a term of one year, another for a term of two years, and another for a term of three years. Thereafter the Board of Directors shall elect from among the Fellows one member annually at their first meeting after their election for a term of three years. The president shall designate one of the Vice-Presidents as Chairman of this Committee.

4. The Institute shall have a Committee on Publications composed of three Members or Fellows elected by the Board of Directors. The President shall designate a Vice-President as Ex Officio Chairman of this Committee.

Article IV

MEETINGS

1. A meeting for the presentation and discussion of papers, for the election of Officers, and for the transaction of other business of the Institute shall be held annually at such time as the Board of Directors may designate. Additional meetings may be called from time to time by the Board of Directors and shall be called at any time by the President upon written request from ten Fellows. Notice of the time and place of meeting shall be given to the membership by the Secretary-Treasurer at least thirty days prior to the date set for the meeting. All meetings except executive sessions shall be open to the public. Only papers accepted by a Program Committee appointed by the President may be presented to the Institute.

2. The Board of Directors shall hold a meeting immediately after their election and again immediately before the expiration of their term. Other meetings of the Board may be held from time to time at the call of the President or any two members of the Board. Notice of each meeting of the Board, other than the two regular meetings, together with a statement of the business to be brought before the meeting, must be given to the members of the Board by the Secretary-Treasurer at least five days prior to the date set therefor. Should other business be passed upon, any member of the Board shall have the right to reopen the question at the next meeting.

3. The Committee on Membership shall hold a meeting immediately after the annual meeting of the Institute. Further meetings of the Committee may be held from time to time at the call of the Chairman or any member of the Committee provided notice of such call and the purpose of the meeting is given to the members of the Committee by the Secretary-Treasurer at least five days before the date set therefor. Should other business be passed upon, any member of the Committee shall have the right to reopen the question at the next meeting.

4. At a regularly convened meeting of the Board of Directors, three members shall constitute a quorum. At a regularly convened meeting of the Committee on Membership, two members shall constitute a quorum.

Article V

PUBLICATIONS

1. In the beginning, the "Annals of Mathematical Statistics" shall serve as the official journal for the Institute. Other publications may be originated by the Board of Directors as occasion arises.

Article VI

EXPULSION OR SUSPENSION

1. Except for non-payment of dues, no one shall be expelled or suspended except by action of the Board of Directors with not more than one negative vote.

Article VII

AMENDMENTS

1. This constitution may be amended by an affirmative two-thirds vote at any regularly convened meeting of the Institute provided notice of such proposed amendment shall have been sent to each Fellow by the Secretary-Treasurer at least thirty days before the date of the meeting at which the proposal is to be acted upon. Voting may be in person or by mail.

BY-LAWS

Article I

DUTIES OF THE OFFICERS, BOARD OF DIRECTORS, COMMITTEE ON MEMBERSHIP, AND COMMITTEE ON PUBLICATIONS

1. The President, or in his absence, one of the Vice-Presidents, or in the absence of the President and both Vice-Presidents, a Fellow selected by vote of the Fellows present, shall preside at the meetings of the Institute and of the Board of Directors. At meetings of the Institute, the presiding officer shall vote only in the case of a tie, but at meetings of the Board of Directors he may vote in all cases. At least three months before the date of the annual meeting, the President shall appoint a Nominating Committee of three members. It shall be the duty of the Nominating Committee to make nominations for Officers to be elected at the annual meeting and the Secretary-Treasurer shall notify all Fellows at least thirty days before the annual meeting. Additional nominations may be submitted in writing, if signed by at least ten Fellows of the Institute, up to the time of the meeting.

2. The Secretary-Treasurer shall keep a full and accurate record of the proceedings at the meetings of the Institute and of the Board of Directors, send out calls for said meetings and, with the approval of the President and the Board, carry on the correspondence of the Institute. Subject to the direction of the Board, he shall have charge of the archives and other tangible and intangible property of the Institute. He shall send out calls for annual dues and acknowledge receipt of same; pay all bills approved by the President for expenditures authorized by the Board or the Institute; keep a detailed account of all receipts and expenditures, prepare a financial statement at the end of each year and present an abstract of the same at the annual meeting of the Institute after it has been audited by a Member or Fellow of the Institute appointed by the President as Auditor. The Auditor shall report to the President.

3. The Board of Directors shall have charge of the funds and of the affairs of the Institute, with the exception of those affairs specifically assigned to the President or to the Committee on Membership. The Board shall have authority to fill all vacancies ad interim, occurring among the Officers, Board of Directors, or in any of the Committees. The Board may appoint such other

committees as may be required from time to time to carry on the affairs of the Institute.

4. The Committee on Membership shall prepare and make available through the Secretary-Treasurer an announcement indicating the qualifications requisite for the different grades of membership.

5. The Committee on Publications, under the general supervision of the Board of Directors, shall have charge of all matters connected with the publications of the Institute, and of all books, pamphlets, manuscripts and other literary or scientific material collected by the Institute. Once a year this Committee shall cause to be printed in the Official Journal the Constitution and By-Laws and a classified list of all the Members and Fellows of the Institute.

Article II

DUES

1. Members shall pay five dollars at the time of admission to membership and shall receive the full current volume of the Official Journal. Thereafter, Members shall pay five dollars annual dues. The annual dues of Fellows shall be five dollars. The annual dues of Sustaining Members shall be fifty dollars. Honorary Members shall be exempt from all dues.

2. Annual dues shall be payable on the first day of January of each year.

3. The annual dues of a Fellow or Member include a subscription to the Official Journal. The annual dues of a Sustaining Member include two subscriptions to the Official Journal.

4. It shall be the duty of the Secretary-Treasurer to notify by mail anyone whose dues may be six months in arrears, and to accompany such notice by a copy of this Article. If such person fail to pay such dues within three months from the date of mailing such notice, the Secretary-Treasurer shall report the delinquent one to the Board of Directors, by whom the person's name may be stricken from the rolls and all privileges of membership withdrawn. Such person may, however, be re-instated by the Board of Directors upon payment of the arrears of dues.

Article III

SALARIES

1. The Institute shall not pay a salary to any Officer, Director, or member of any committee.

Article IV

AMENDMENTS

1. These By-Laws may be amended in the same manner as the Constitution or by a majority vote at any regularly convened meeting of the Institute, if the proposed amendment has been previously approved by the Board of Directors.

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